A Delzant-Type Correspondence for Hamiltonian SO(3) Actions

Theodore Mollano
Williams College

Advisor: Professor Ivo Terek

Abstract

We establish a Delzant-type classification theorem for compact connected symplectic manifolds equipped with Hamiltonian SO(3) actions, analogous to the construction of a Hamilitonian Delzant space $T^n \circlearrowright M_\Delta$ associated to each polytope Δ, as was done in Delzant's Thesis.

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Abstract

In Delzant's Thesis, Delzant (1988), establishes a one-to-one correspondence between compact connected symplectic manifolds with effective Hamiltonian toric actions, and Delzant polytopes. To each Delzant polytope Δ, we associate a Delzant Space M_Δ, with Hamilitonian toric action $T^n \circlearrowright M_\Delta$. The moment map corresponding to this toric action realizes Δ as the image of M_Δ via $\mu(M_\Delta) = \Delta$. In this thesis, we are interested in a generalization of Delzant's Theorem to non-Abelian case of a SO(3) Lie group action. We revisit a result by Blaom (1996), which states that for Hamiltonian SO(3)-spaces, in the case where the moment image avoids the origin, two such spaces are equivalent if and only if their associated maximal torus spaces are equivalent. We provide a new reformulation of this result, and an associated classification scheme of the moment images. We also that the regular momenta condition is necessary by examining a natural SO(3)-action on $\mathbb{C}P^2$.

Executive Summary: Delzant-Type Correspondence For Hamiltonian SO(3)-actions

Background and Context

Although classical mechanics was once done using Newtonian Mechanics $m\ddot{x} = \sum_i \vec{F}_i$, it is more elegantly formulated in terms of Hamiltonian mechanics, which describes the time evolution of position q^i and momentum p_i of a solid body via the dynamics of a scalar function, called the Hamiltonian $H(q,p)$. This occurs via the set of differential equations:

$$H \text{ determines dynamics via} \qquad \dot{q}^i = \frac{\partial H}{\partial p_i} \text{ and } \dot{p}_i = -\frac{\partial H}{\partial q^i} \qquad \text{(Hamilton's Equations)}$$

This approach to classical mechanics is readily formulated on manifolds via Hamiltonian vector fields. When additional structure is added to the manifold, such as a Hamiltonian Lie group action G, we define a function μ, the moment map, which captures the conserved quantities associated with symmetry induced by $G \circlearrowleft M$. For toric manifolds with actions of the torus group T^n, Delzant's theorem provides a complete classification: there exists a one-to-one correspondence between compact connected symplectic manifolds with effective Hamiltonian T^n actions and Delzant polytopes in $\mathbb{R}^n$, and this correspondence is given by the moment map $\mu(M_\Delta) = \Delta$.

Research Problem

While Delzant's theorem provides a complete classification for Abelian group actions, Guillemin (1994) identified the situation for non-Abelian groups as complex and less understood.

Key Results

Our main result establishes that for Hamiltonian SO(3) spaces with regular momenta (where the moment map avoids the origin), there exists a classification similar to Delzant's theorem. Specifically, based on Blaom (1996) we show:

1. Two Hamiltonian SO(3)-spaces with moment map image avoiding the origin are equivalent if and only if their associated maximal torus ($T \cong S^1$) spaces are equivalent.
2. For 4-dimensional compact Hamiltonian SO(3) spaces whose moment images do not contain the origin, if the rotation actions in S^1_{xy} are faithful on the preimage of the z-axis, then these spaces are equivalent up to equivariant symplectomorphism.

Classification Results

We provide a complete classification of possible Hamiltonian SO(3)-actions on symplectic manifolds:

Dim	**Faithful**	**Manifold**	**Action**	**Moment Image**
2	Y	$S^2(r)$	$A \cdot p = Ap$	Sphere $S^2(r)$
4	Y	$S^2(r_1) \times S^2(r_2)$	$A \cdot (p, q) = (Ap, Aq)$	Hollow ball
4	Y	$S^2(r) \times S^2(r)$	$A \cdot (p, q) = (Ap, Aq)$	Closed ball
2	N	S^2	$A \cdot p = p$	Origin $\{0\}$
4	Y	$S^2(r) \times S^2$	$A \cdot (p, q) = (Ap, q)$	Sphere $S^2(r)$

Necessity of Regular Condition

Regular momenta is a requirement for this correspondence to hold. We demonstrate this through a counterexample: the SO(3)-action on $\mathbb{CP}^2$ produces a closed ball as its moment image, which suggests it is equivalent to $S^2(r) \times S^2(r)$. However, these manifolds have different cohomology groups: $H^2_{\mathrm{dR}}(\mathbb{CP}^2) = \mathbb{R}$ vs. $H^2_{\mathrm{dR}}(S^2 \times S^2) = \mathbb{R}^2$, proving they cannot be equivalent.

Implications and Future Work

Our results extend Delzant's fundamental work to the non-Abelian setting, which is a step towards understanding Hamiltonian group actions beyond the toric case. Future directions include:

1. Determining which Melvin-Parker manifolds admit symplectic forms compatible with SO(3)-actions.
2. Extending the classification to other non-Abelian Lie groups.

Acknowledgments

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Chapter 1

Introduction

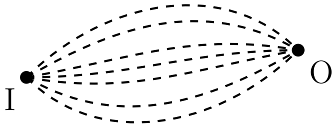
1.1 Introduction to Symplectic Geometry

The goal of this thesis is to present a proof of a Delzant Correspondence for Hamiltonian SO(3) actions. This introduction answers three fundamental questions:

1. What is a meaningful reformulation of classical mechanics?
2. What is a moment map?
3. Why is one interested in Delzant's Theorem and its generalizations?

1.1.1 Motivation for Symplectic Geometry

In 15th century CE, Fermat discovered the least path principle for light, which dictates that light will always travel on a path such that its travel time is a local minimum in the space of possible paths. The index of refraction of a material, $\beta \in \mathbb{R}^+$, is a scalar quantity which is the factor by which light slows in a material, $v_{\text{eff}} = c/\beta$. Fermat's principle explains Snell's law, which states for two media with indices of refraction β_1 and β_2, a light ray striking the β_1-indexed material at an angle θ_1 from the normal will scatter at an angle θ_2. The scattering angle θ_2 is determined by the three parameters β_1, θ_1 and β_2 via $\beta_1 \sin\theta_1 = \beta_2 \sin\theta_2$. Snell's law revolutionized the way mechanics was done.



$\tau = \int_{\eta(t)} \frac{1}{\beta} dt$ is minimized

This principle was generalized to the Lagrangian Principle, which states that for a mechanical system, described by position $(q^1, \ldots, q^n)$ and velocity $(\dot{q}^1, \ldots, \dot{q}^n)$, we can construct a scalar function $\mathcal{L}(q^i, \dot{q}^i)$ called the Lagrangian, such that its integral, called the action, is stationary for $\eta(t)$ in the space of possible paths.

$$\delta S = \delta \left[\int_{\eta(t)} \mathcal{L} dt \right] = 0 \qquad \text{(The Variational Principle)}$$

One may show this condition is equivalent to $\mathcal{L}$ satisfying the following differential equation:

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0 \text{ for all coordinates } q^i \qquad \text{(The Euler-Lagrange Equations)}$$

Now, suppose we are given a scalar quantity, H called the Hamilitonian, such that:

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \text{and} \quad \dot{q}^i = \frac{\partial H}{\partial p_i} \tag{1.1}$$

Consider the expression:

$$\mathcal{L} = \sum_i p_i \dot{q}^i - H \tag{1.2}$$

Applying the definitions, it follows that:

$$\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{\partial H}{\partial q_i} \text{ and } \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i \text{ imply} \qquad \frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0 \qquad \text{(A New Formula for } \mathcal{L}\text{)}$$

Therefore, 1.2 is the Lagrangian for a mechanical system. The action S for this mechanical system is then written: $S = \int_t p_i dq^i - H dt$. Symplectic geometry was invented to study the term $\theta = p_i dq^i$ in the integral of S. Namely, the symplectic form, the primary tool of symplectic geometry, is defined to be $\omega = -d\theta$.

Remark 1.1.1. *From the integral based intuition for ω - understanding ω is equivalent to understanding how the integral S behaves*

References: Arnold (2010), Terek (2020), Feynman, Leighton, and Sands (2010)

1.1.2 Delzant's Correspondence Theorem

In this thesis, we describe how a scalar function generates vector fields One structure that we impose is a Lie group action on our manifold. Now, when a Lie group G acts on a manifold M, it establishes an orbit map $\sigma_p : G \to M$. This map can be differentiated at the identity of G, giving rise to a linear transformation $(d\sigma_p)_e : \mathfrak{g} \to T_pM$, where $\mathfrak{g}$ is the Lie algebra of G. Define the action field corresponding to $X \in \mathfrak{g}$ to be $X_p^{\#} = (d\sigma_p)_e X$. For each vector

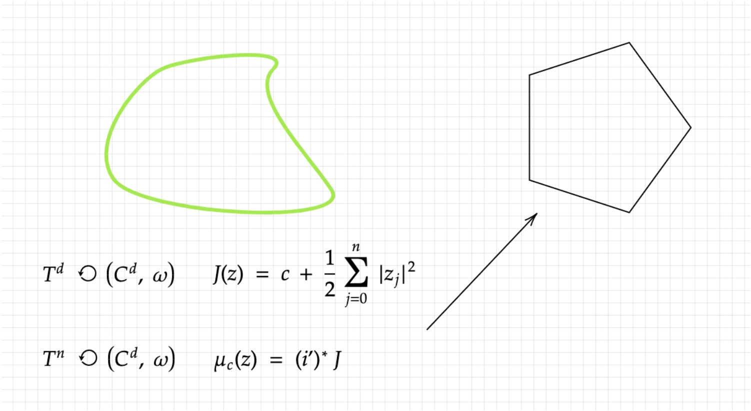


Figure 1.1: Identifying polytopes Δ as images of moment maps via Delzant's Theorem.

field, we ask the question: Does there exist a function which generates this vector field? These functions are known as moment maps and are crucial to the symplectic framework. One classical result in symplectic geometry is Delzant's Correspondence Theorem. It states that there is a one-to-one correspondence between symplectic toric manifolds and Delzant polytopes — the latter arise as images of moment maps associated with the former. Denote a polytope Δ. Then its corresponding Delzant space is (X_Δ, ω). Delzant's Theorem is quite interesting because it establishes that the images of momentum maps themselves are geometric objects. Understanding what happens for more general Hamiltonian G-spaces (where G is any compact and connected Lie group, not necessarily the torus) is one of the most challenging problems in symplectic geometry.

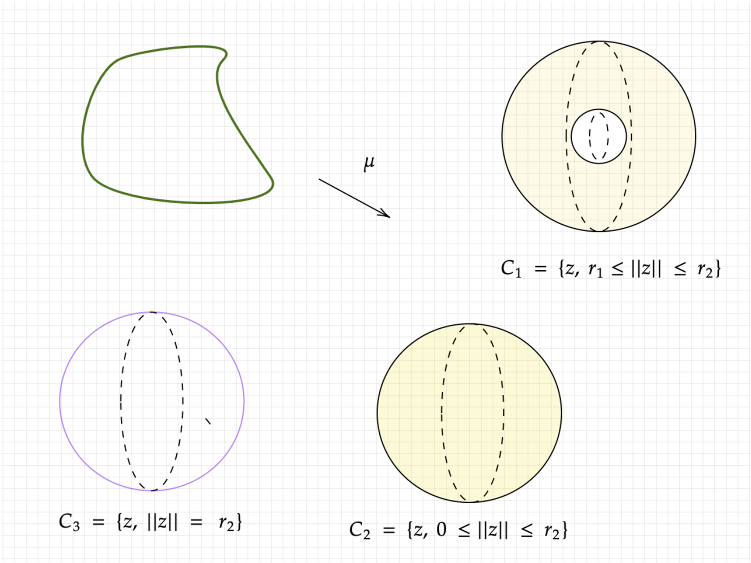


Figure 1.2: Possibilities of nontrivial moment-images when $\mathrm{SO}(3) \circlearrowright M$.

Now, our investigation starts with the simplest compact and connected non-Abelian group:

the special orthogonal group SO(3). The moment-images for (compact and connected) Hamiltonian SO(3)-spaces, which are necessarily SO(3)-invariant subsets of $\mathfrak{so}(3)^* \cong \mathbb{R}^3$, can then only fall into one among four cases: the origin, a sphere, a closed ball, or a closed hollow ball. It is easy to construct examples showing that all four possibilities do occur, but to which precise extent does this image determine the original space? What is a Delzant correspondence for Hamiltonian SO(3)-spaces, if there is even one at all?

References: Atiyah, Guillemin, and Sternberg (1982), Delzant (1988), Blaom (1996)

Chapter 2

Background

2.1 Symplectic Manifolds and Hamiltonian Actions

In this section, we will see the prototypical examples for symplectic manifolds, and discuss with more precision the definitions that guide the subject. This section will answer the questions:

1. How does one define classical Mechanics?
2. What is a symplectic manifold and what are its technicalities?
3. How does one define rigorously a Lie group action?
4. How to define Hamiltonian actions?

2.1.1 Classical Mechanics

Classical mechanics studies rigid bodies. Symplectic geometry is modeled after classical mechanics. Therefore, we define several notions in classical mechanics. Each classical mechanical system has an underlying geometry Q. This is where motion takes place:

Definition 2.1.1 (Configuration Space of a Mechanical System)**.** *The configuration space of a mechanical system is a smooth manifold* Q*, called the configuration manifold.*

Furthermore, for motion to occur, we require both position and momentum to be well defined:

Definition 2.1.2 (Positions and Momenta)**.** *Given a configuration space* Q*, the position* $q = (q^1, \dots, q^n)$ *of a mechanical element is a coordinate pair made up of smoothly valued chart functions on* Q*. Given a chart* (U, q) *on* Q*, we have that* $q : U \to \mathbb{R}^k$*, where* $k = \dim U$*. Momenta of the mechanical system on the other hand are are linear maps on the cotangent space:* $(T_xQ)^*$*. Identifying basis vectors of the cotangent space* $(T_xQ)^*$ *in charts as* $\frac{\partial}{\partial q^i}$*, we have that:* $p = \sum_i p_i \frac{\partial}{\partial q^i}$ *and* $p : T_xU \to \mathbb{R}^k$*.*

One key distinction in symplectic geometry versus classical Newtonian mechanics is that in symplectic geometry, when we work on a configuration space Q, position coordinates are valued in Q, while momentum vectors to $x \in Q$ are valued in $(T_xQ)^*$, the dual space to T_xQ. The reason for this distinction comes from Riemannian geometry. When our manifold Q has a Riemannian metric g, we will show that momentum is valued in $(T_xQ)^*$, where $x \in Q$.

Definition 2.1.3 (Kinetic Energy using g)**.** *Given a Riemannian manifold (Q, g), kinetic energy E is an inner product valued on its tangent space T_xQ. The inner product is:*

$$E = T_xQ \times T_xQ \to \mathbb{R},\ E = \frac{m}{2}\langle \cdot, \cdot \rangle$$

Remark 2.1.1. *Let $q, \dot{q}$ be the position and velocity of a particle. This definition of kinetic energy generalises the formula $E = \frac{m}{2}|\dot{q}|^2$ to configuration space Q. In classical mechanics, it is also common to write $E = \frac{1}{2}pv$. We require a new definition of momentum, p.*

Definition 2.1.4 (Momentum as a Linear Map)**.** *Define momentum to be: $p = \langle \dot{q}, \cdot \rangle$. Therefore, $p \in T_xQ^*$, and $E = \frac{1}{2}p(\dot{q})$*

Remark 2.1.2. *When (Q, g) is a Riemannian manifold, it is clear that for $x \in Q$, momentum is $p \in (T_xQ)^*$. This definition also allows us to define momentum even if the configuration space Q does not possess a Riemannian metric.*

Definition 2.1.5 (Phase Space of a Mechanical System)**.** *The phase space of a mechanical system is the manifold which contains all of the information necessary to describe mechanical motion. The phase space of classical mechanical system is T^*Q:*

$$T^*Q = \bigsqcup_{q \in Q} T_q^*Q = \{(q, d\phi) \in T_qQ^*\}$$

Lemma 2.1.1. *The phase space of a classical mechanical system is a vector bundle $\pi : T^*Q \to Q$, where $\pi(q, d\phi) = q$*

Proof. For a chart (x, U) on Q, we must define a bundle chart, $h_x : \pi^{-1}(U) \to U \times \mathbb{R}^n$. The following choice of bundle chart suffices:

$$h_x : \pi^{-1}(U) \to U \times (\mathbb{R}^n)^* \cong U \times (\mathbb{R}^n) : (q, d\phi) \to (q, D(\phi x^{-1}) \circ x(q))$$

using the definition of $d\phi$ in coordinates. For the next isomorphism, we use the canonical isomorphism between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$. In addition, it must be shown that the map h_x also induces an isomorphism between fibres $\pi^{-1}(q) \cong \mathbb{R}^n$. This occurs as the map $D(\phi x^{-1}) \circ x(q)$ has full rank. Therefore, for each $p \in Q$, there is an open set U, a homeomorphism $h : \pi^{-1}U \to U \times \mathbb{R}^n$ such that

$$\begin{tikzcd}
 & U \times \mathbb{R}^n \arrow[d, "\pi"] \\
\pi^{-1}(U) \arrow[ur, "h"] \arrow[r, "\pi|_{\pi^{-1}(U)}"] & U
\end{tikzcd}$$

Such that the diagram commutes, and the composite map: $h_q : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector space isomorphism. □

Definition 2.1.6 (Definition of a Mechanical System)**.** *A mechanical system (Q, H) is a configuration space Q, a phase space T^*Q, and a smooth scalar function on the phase space $H : T^*Q \to \mathbb{R}$. Time evolution of the mechanical system is governed by the Hamilton Equations:*

$$\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q}$$

Alternatively, we can write them in each coordinate as: $\dot{q}^i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q^i}$.

Lemma 2.1.2 (Model Space for Symplectic Geometry)**.** *Consider the mechanical system (Q, H), and its phase space T^*Q. Consider the vector field:*

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{2.1}$$

Define the symplectic form ω, on T^*Q to be:

$$\omega = \sum_{i \geq 1} dq^i \wedge dp_i$$

Then we have the following: $\omega(X_H, \cdot) = dH$.

Proof. Writing out the evaluation:

$$\omega(X_H, \cdot) = \sum_{i \geq 1} \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \right) \left[dq^i \wedge dp_i \right] = \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial q^k} dq^k = dH$$

The last step follows from the chain rule, appLied to $H : T^*Q \to \mathbb{R}$. □

Remark 2.1.3. *This reformulation of Hamilton's equations also works in the reverse direction.*

Now, we would like to show the opposite direction. To do so, we want to be able to invert $\omega(X_H, \cdot) = dH$. Therefore, we make the following definition:

Definition 2.1.7. *A symplectic form ω is non-degenerate if when vector field Z satisfies $\omega(Z, X) = 0$ for all $X \in \mathfrak{X}(T^*Q)$, then $Z = 0$ everywhere. Namely, if ω is non-degenerate, then for any $x \in Q$*

$$\omega_x^{-1} : (T_xQ)^* \to T_xQ \textit{ given by } [\omega(v, \cdot) \to v] \textit{ is an isomorphism.} \tag{2.2}$$

Remark 2.1.4. *When a symplectic form is non-degenerate, this has the unique consequence that its inverse, ω^{-1} is well-defined and therefore unique.*

Lemma 2.1.3 (Non-degeneracy of ω)**.** *Given a configuration space Q, for its phase space T^*Q, the symplectic form:*

$$\omega = \sum_{i \geq 1} dq^i \wedge dp_i$$

is non-degenerate.

Proof. Suppose a vector field Z satisfies $\omega(Z, X) = 0$ for all $X \in \mathfrak{X}(T^*Q)$. Let $Z = \alpha(q,p)\frac{\partial}{\partial q^i} + \beta\frac{\partial}{\partial p_i}$. Then, we choose $X^1 = \frac{\partial}{\partial q^i}$ and $X^1 = \frac{\partial}{\partial p_i}$. We note that:

$$\omega(Z, X^1) = \alpha(q,p) = 0 \qquad\qquad \omega(Z, X^2) = \beta(q,p) = 0 \tag{2.3}$$

Therefore, $Z = 0$ and ω is non-degenerate ☐

Definition 2.1.8 (Integral Curve on T^*Q)**.** *Consider the configuration manifold Q, and its phase space T^*Q. Given a vector field $X_H \in \mathfrak{X}(T^*Q)$, the integral curve of that vector field is $\gamma(t) = (\gamma_1, \ldots, \gamma_n)$, such that: $\dot{\gamma}_j = X_H(\gamma)_j$.*

Lemma 2.1.4 (Reverse Implication)**.** *Consider the mechanical system (Q, H), and its phase space T^*Q. Let ω be defined as above, such that: $\omega(X_H, \cdot) = dH$. Then the integral curve of X_H, $\gamma(t)$ satisfies Hamilton's equations.*

Let: $\gamma(t) = (q^1(t), \ldots, q^n(t), p_1(t), \ldots, p_n(t))$ be the integral curve of X_H. Since ω is non-degenerate, its inverse is unique. So, we have that X_H has the same definition as in 2.1.2.

Proof. Now, according to the definition of an integral curve: $\dot{\gamma}_j = [X_H(\gamma)]_j$
Matching coefficients, we have:

$$\dot{\gamma}_i \frac{\partial}{\partial q^i} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} \qquad\qquad \dot{\gamma}_{i+n}\frac{\partial}{\partial p_i} = -\frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{2.4}$$

Unraveling the definition $\dot{\gamma}_i = \frac{dq^i}{dt}$ and $\dot{\gamma}_{i+n} = \frac{dp_i}{dt}$, we have that:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{2.5}$$

Therefore, the equations of motion of a classical mechanical system (Q, H) and the equation $\omega(X_H, \cdot) = dH$ on T^*Q are equivalent. ☐

Remark 2.1.5. *A generalization of classical mechanics would to be study a manifold M, with a closed, non-degenerate two form ω, a smooth function $H : M \to \mathbb{R}$, and vector fields X_H such that $\omega(X_H, \cdot) = dH$. This would mimic a classical system and replicate it in the case that $M = T^*Q$, and $\omega = \sum_{i \geq 1} dq^i \wedge dp_i$. This generalization of classical mechanics is symplectic geometry and will be developed in the following section.*

References: Dundas (2018), Terek (2020), Arnold (2010)

2.1.2 Symplectic Structures

As in 2.1.5, symplectic geometry studies manifolds equipped with a two form (M, ω) which have the model geometry of T^*Q. As usual in differential geometry, for this construction we must describe objects on the linear level before moving to the global level.

Definition 2.1.9 (Symplectic Vector Space)**.** *A symplectic vector space V is a pairing (V, Ω), where V is a vector space, and $\Omega : V \times V \to \mathbb{R}$ is a non-degenerate skew-symmetric bilinear form. Here, non-degeneracy means that $\Omega(v, w) = 0$ for all $w \in V$ if and only if $v = 0$.*

Remark 2.1.6. *The symplectic prototype resembles T^*Q, as that geometry came equipped with another skew-symmetric, bilinear form, namely ω.*

Remark 2.1.7 (A Bilinear Form is just a Skew-Symmetric Matrix)**.** *A non-degenerate skew-symmetric bilinear form is just a skew-symmetric matrix, when we fix a choice of basis. Therefore, we can take the prototypical symplectic manifold to have the off-diagonal skew-symmetric identity matrix.*

Example 2.1.1 (Symplectic Vector Space Prototype of $\mathbb{R}^{2n}$)**.** *The prototypical symplectic vector space is the vector space $(\mathbb{R}^{2n}, \Omega = \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix})$ with the standard bases. One computes $\Omega(v, w)$ according to the following prescription: $\omega(v, w) = v^T \Omega w \in \mathbb{R}$.*

Lemma 2.1.5 (Vector Space Prototype is Well-defined)**.** *The symplectic vector space prototype is non-degenerate, bilinear and skew-symmetric.*

Proof. The expression $\big[(v, w) \to v^T \Omega w\big]$ is bilinear, by matrix composition of the row space, versus the column space. Now, to show the symplectic prototype Ω is skew-symmetric. Consider the standard basis for $\mathbb{R}^{2n}$, that is $\mathcal{B} = \{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots (0, 0, \ldots, 1)\}$ written as:

$$\mathcal{B} = \{e_1, \ldots e_n, h_1, \ldots, h_n\}$$

We note that on bases, Ω is:

$$\Omega(e_j, h_j) = 1 \qquad \Omega(h_j, e_j) = -1 \qquad \Omega(h_i, h_j) = \Omega(e_i, e_j) = 0 \tag{2.6}$$

Therefore, Ω is skew-symmetric on bases. By bilinearity, it is skew symmetric on $V \times V$. Additionally, as Ω has determinant 1, it establishes an isomorphism $\Omega^{-1} : V^* \to V$, and we note that as a result, Ω is non-degenerate. □

Remark 2.1.8. *To build a symplectic manifold, the generalization of classical mechanics based on the model geometry T^*Q, we want each of the tangent spaces to a manifold M, to be isomorphic the symplectic prototype, and the symplectic form on our manifold M, ω to be the prototype symplectic form in each tangent space as well.*

Remark 2.1.9. *In 1.1.1, we took an approach to the symplectic form in a different way, where we defined it as negative the exterior derivative of the symplectic potential θ. From this intuition, we require another condition on ω, which is that it is closed. This is because the exterior derivative operator satisfies $d^2 = 0$. In general, whenever a two form is closed, we can always write it as an exterior derivative of a one form locally, so the symplectic potential will always make sense. To understand this further see: Terek (2020).*

Definition 2.1.10 (Definition of a Symplectic Manifold)**.** *A symplectic manifold (M, ω) is a smooth manifold M, equipped with a two form $\omega \in \Omega^2(M)$, that is closed, non-degenerate, and skew-symmetric.*

Example 2.1.2 (Sympletic Geometry of $(\mathbb{C}^n, \omega)$)**.** *Let $z = (z_1, \dots, z_n)$, where each $z_j = x_j + iy_j$. Making note of the isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$, there exists the non-degenerate, skew-symmetric, bilinear form inherited from $\mathbb{R}^{2n}$ 2.1.1. Then: $\omega_{\mathbb{R}^{2n}} = dx_1 \wedge dy_1 + \cdots + dx_2 \wedge dy_n$. It is an observation that:*

$$d\overline{z}_j \wedge dz_j = [dx_j - idy_j] \cdot [dx_j + idy_j] = idx_j \wedge dy_j$$

Therefore, we define $\omega_{\mathbb{C}^n} = \frac{-i}{2} \sum_{j \geq 1} \overline{dz}_j \wedge dz_j$.

Example 2.1.3 (Sympletic Geometry of a Surface in $\mathbb{R}^3$)**.** *Let $M \subset \mathbb{R}^3$ be a orientable surface, M is an embedded manifold in $\mathbb{R}^3$. Since M is an orientable surface, it admits a normal vector field $x \to N_x$. Therefore, a skew-symmetric form on M is the area form of M: $\omega_x(v, w) = \langle N_x, v \times w \rangle$.*

Lemma 2.1.6 (Surface Form)**.** *The area form on a surface ω 2.1.3 is a skew-symmetric, non-degenerate, bilinear form*

Proof. Let's prove ω is non-degenerate. Let $v, w \in \mathfrak{X}(M)$. Let v_x, w_x be the tangent vectors at $x \in M$. Suppose $\omega(v, w) = 0$ for all $w \in \mathfrak{X}(M)$. Then:

$$\langle N_x, v_x \times w_x \rangle = 0$$

Since $\langle \cdot, \cdot \rangle$ is non-degenerate, and $N_x \neq 0$, we have for all $w_x \in T_xM$

$$v_x \times w_x = 0$$

Choose w_x perpendicular to v_x in the plane of T_xM. Then, $v_x = 0$. Therefore, $v = 0 \in \mathfrak{X}(M)$. It follows from an observation that ω is skew-symmetric. We observe that:

$$\omega(v, w) = -\omega(w, v)$$

Finally, since the cross product is bilinear, we conclude. □

References: Terek (2020), Dundas (2018)

2.1.3 No Local Invariants of Symplectic Manifolds

It is important to see how diffeomorphisms alter the symplectic form. As usual in differential geometry, we study the case when diffeomorphisms preserve the symplectic form:

Definition 2.1.11 (Pullback of a symplectic form)**.** *Suppose we are given a symplectic manifolds* (M', ω') *and* (M', ω')*, vectors* $v, w \in T_xM$*, and a point* $x \to \phi(x) = x'$*. If we have a diffeomorphism* $\phi : M \to M'$*, the pullback of* ω'*, denoted* $\phi^*\omega'$ *is the map:*

$$\phi^*\omega'_{x'}(v, w) = \omega'_{x'}(d\phi_x v, d\phi_x w)$$

Definition 2.1.12 (Symplectomorphism)**.** *Given two symplectic manifolds* (M, ω) *and* (M', ω')*, a symplectomorphism* ϕ *between them is a diffeomorphism* ϕ *such that for any point* $x \in M$*,* $v, w \in T_xM$*:*

$$\left[\phi^*\omega'_{x'}\right](v, w) = \omega_x(v, w)$$

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[d, "x\to \omega_x"'] & M' \arrow[d, "x'\to \omega'_{x'}"] \\
\Lambda^2(T^*M) & \Lambda^2(T^*M') \arrow[l, "\phi^*"]
\end{tikzcd}$$

A symplectomorphism is a map ϕ *which makes the following diagram commute.*

In each of the above examples 2.1.3 and 2.1.2, the symplectic forms all were (up to relabeling):

$$\omega = \sum_{j \geq 1} dq^j \wedge dp_j$$

Remark 2.1.10 (No Local Invariants)**.** *Was this by accident? One observation early on in symplectic geometry was that all symplectic manifolds are in fact locally symplectomorphic, we will see this in the following theorem 2.1.7.*

Theorem 2.1.7 (Darboux's Theorem – All Symplectic Manifolds are Locally Symplectomorphic)**.** *Let* (M, ω) *be a symplectic manifold, and* $p \in M$ *be any point. Then, there is a chart* $(U, (x^1, \ldots, x^n, y^1, \ldots, y^n))$ *around* p *for which:* $\omega = \sum_{j \geq 1} dx_j \wedge dy_j$ *All symplectic manifolds locally look the same, and there are no local invariants in symplectic geometry.*

For a proof of 2.1.7, see Arnold 2010. This is an important result, since it allows us to take the canonical local coordinates when we need to do so.

2.1.4 Algebras of Vector Fields

We defined a symplectic manifold (M, ω) based on the model geometry of the bundle: $T^*Q \to Q$. On the manifold T^*Q, we reformulated classical mechanics in terms of the symplectic form there 2.1.2.

Remark 2.1.11. *This reformulation of classical mechanics can also be done for a symplectic manifold* (M, ω)*.*

The generalization of Hamilton's equations becomes the definition of a Hamilitonian vector field.

Definition 2.1.13 (Hamiltonian Vector Field)**.** *Let* (M, ω) *be a symplectic manifold, then, a vector field* $X_H \in \mathfrak{X}(M)$ *is called Hamiltonian if there exists a smooth Hamilitonian function* $H : M \to \mathbb{R}$ *such that* $\omega(X_H, \cdot) = dH$*.*

Since ω is non-degenerate, ω^{-1} exists and $X_H = \omega^{-1} dH$. A weaker notion than a Hamilitonian vector field is a symplectic vector field. Such a vector field does not necessarily derive from a Hamilitonian function H.

Definition 2.1.14 (Symplectic Vector Field)**.** *Let* (M, ω) *be a symplectic manifold, then, a vector field* $X \in \mathfrak{X}(M)$ *is called symplectic if* $\omega(X, \cdot)$ *is closed.*

Lemma 2.1.8 (Inclusion Law)**.** *Every Hamilitonian vector field is a symplectic vector field.*

Proof. Note that $\omega(X_H, \cdot) = dH$, which is closed. □

Remark 2.1.12. *Symplectic vector fields and Hamilitonian vector fields form algebras with composition laws. To see these, we first define function composition on our symplectic manifold. Function composition occurs via the Poisson-bracket* $\{\cdot, \cdot\}$

Definition 2.1.15 (Poisson Bracket)**.** *Consider a symplectic manifold* (M, ω)*. Let* f, g *be two smooth functions* $f, g : M \to \mathbb{R}$*, and* X_f *and* X_g *be the corresponding Hamiltonian vector fields. The Poisson bracket between* f *and* g *is a map:*

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \qquad \{f, g\} = \omega(X_f, X_g) \tag{2.7}$$

Lemma 2.1.9 (Lie Bracket is the Poisson Bracket)**.** *Given Hamilitonian vector fields* X_f *and* X_g*, as defined in 2.1.15, we have that:*

$$[X_f, X_g] = X_f X_g - X_g X_f = -X_{\{f,g\}}$$

Proof. Lemma is proved in appendix: A.2.3 □

Definition 2.1.16 (Algebras of Vector Fields)**.** *Consider a symplectic manifold* (M, ω)*. Let* $\mathcal{S}(M, \omega, [\cdot, \cdot])$ *and* $\mathcal{H}(M, \omega, [\cdot, \cdot])$ *denote the algebras of Symplectic and Hamiltonian vector fields on* M

Lemma 2.1.10 (Sym Algebra is Well-Defined)**.** *Let (M,ω) be a symplectic manifold as above. We first show that $\mathcal{S}(M,\omega,[\cdot,\cdot])$ is closed under the Lie-bracket $[\cdot,\cdot]$.*

Proof. Note that if $X, Y \in \mathfrak{X}(M)$ is symplectic, $\omega(X,\cdot)$ is closed. For some $x \in M$, consider a small neighborhood V, where $x \in V$. Since X is symplectic, we write: $\omega(X,\cdot) = df'$, where $df' : V \to \mathbb{R}$ is a smooth function. We do the same for Y, writing $\omega(X,\cdot) = dg'$. Therefore, we understand that X and Y are locally Hamilitonian, and so we write: $[X,Y] = -X_{\{f',g'\}}$. It suffices to note that $\{f',g'\}$ now is the symplectic function for $[X,Y]$. $\square$

Lemma 2.1.11 (Ham Algebra is Well-Defined)**.** *Let (M,ω) be a symplectic manifold as above. We show that $\mathcal{H}(M,\omega,[\cdot,\cdot])$ is closed under the Lie-bracket $[\cdot,\cdot]$.*

Proof. This proof follows from the above argument for $\mathcal{S}(M,\omega,[\cdot,\cdot])$, noting that now when f, g are functions $M \to \mathbb{R}$, $\{f,g\}$ is also a function $M \to \mathbb{R}$ $\square$

References: Terek (2020), Dundas (2018), Arnold (2010)

2.1.5 Lie Group Actions

One natural question in manifold theory is whether a Hamilitonian H can generate a symmetry from its dynamics.

Remark 2.1.13. *These are made explicit by having a Lie group G, act on the manifold. The Lie group action, establishes a method of studying the action of symmetry.*

Definition 2.1.17 (Lie Group Action)**.** *Let G be a Lie group, and M be a manifold. A smooth, left action of G on M, denoted $G \circlearrowright M$ is a smooth map: $G \times M \to M$ given by $(g,x) \mapsto g \cdot x$ satisfying $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$.*

Example 2.1.4 (Toric Action $T^n \circlearrowright \mathbb{C}^n$)**.** *One common Lie group action is that of the torus, T^n acting on $\mathbb{C}^n$:*

$$(e^{i\beta_1}, \ldots, e^{i\beta_n}) \cdot (e^{i\theta_1}, \ldots, e^{i\theta_n}) = (e^{i(\theta_1+\beta_1)}, \ldots, e^{i(\theta_n+\beta_n)}) \tag{2.8}$$

The toric action is given by phase rotation in each of the complex coordinates.

To each Lie group element G, there exists an associated mapping:

Definition 2.1.18 (Orbit Map)**.** *Consider a symplectic manifold (M,ω) with a Lie-group action $G \circlearrowright M$. Given any point $x \in M$, we define the orbit map $\sigma_x : G \to M$ to be the map $[g \to g \cdot x]$.*

Definition 2.1.19 (Classifying Lie Group Actions)**.** *Let $G \circlearrowright M$ be a Lie group action on a manifold:*

1. *Given $x \in M$, $G \cdot x = \{g \cdot x : g \in G\}$ is the orbit of x.*

2. *Given $x \in M$, $G_x = \{g \in G : g \cdot x = x\}$ is the stabilizer of x.*
3. *The action is transitive if there is an $x \in M$ such that $G \cdot x = M$.*
4. *The action is free if $G_x = e$ for all $x \in M$.*
5. *The action is proper if the enriched action $G \times M \to M \times M$ is a proper map.*
6. *The action is effective if $g \cdot x = x$ for all $x \in M$ impLies that $g = e_G$.*

Since a Lie-group action $G \circlearrowleft M$ is smooth, the orbit map is differentiable.

Definition 2.1.20 (Lie Algebra of G, $\mathfrak{g}$)**.** *The Lie Algebra of G, $(\mathfrak{g}, [\cdot,\cdot])$ is the tangent space to G at the identity, e: $\mathfrak{g} = T_eG$. The composition rule in $\mathfrak{g}$ is given by $[\cdot,\cdot]$, which is the Lie bracket on $\mathfrak{g}$.*

Consider the conjugation map: $\Psi_g : G \to G$ given by:

$$\Psi_g(h) = ghg^{-1}$$

Since the action of $G \circlearrowleft G$ is smooth, we take its derivative at the identity:

$$\left(d\Phi_g(X)\right)_e : T_eG \to T_eG$$

Then, the map: $g \to \left(d\Phi_g(X)\right)_e$ can be differentiated at the identity.

$$\mathrm{ad}_X(Y) : T_eG \times T_eG \to \mathbb{R} \qquad\qquad \mathrm{ad}_X(Y) = [X, Y] \tag{2.9}$$

Definition 2.1.21. *Lie bracket on G, $[\cdot,\cdot]$ The Lie-bracket is the quantity:*

$$\mathrm{ad}_X(Y) = [X, Y]$$

The Lie bracket will be important in the study of the Lie algebra of $\mathfrak{so}(3)$. Since $G \circlearrowleft M$ is smooth, the derivative of the orbit map at the identity is well defined:

Definition 2.1.22 (Derivative of the Orbit Map)**.** *Given a symplectic manifold (M, ω), and a Lie group action $G \circlearrowleft M$, and a point $x \in M$, the orbit map σ_x can be differentiated at the identity to yield the map:*

$$\left(d\sigma_x\right)_e : \mathfrak{g} \to T_xM$$

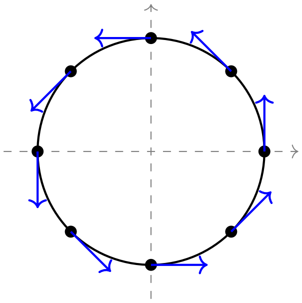
Definition 2.1.23 (Fundamental Vector Field)**.** *Let $X \in T_eG$. The vector field $X^\#$ is the image of X, as x ranges over M:*

$$X^\# = \bigsqcup_{x \in M} \left(d\sigma_x\right)_e (X)$$

Example 2.1.5 (Action of $S^1 \circlearrowright S^1$, What is $X^\#_{\theta_1}$)**.** *Consider a symplectic manifold* (S^1, ω)*, acted on by a Lie-group action*

$$S^1 \circlearrowright S^1 \; \textit{via} \; [e^{i\theta} \cdot e^{i\theta_1} \to e^{i(\theta+\theta_1)}] \tag{2.10}$$

In this case, $\sigma_x(\theta) = e^{i\theta}x$*, and* $(d\sigma_x)_e = ix$*. Since* $ix = X^\#_{\theta_1}(x)$*, we deduce that* $X^\#_{\theta_1} = \frac{\partial}{\partial\theta_1}$*.*



Remark 2.1.14. *We want to study which Hamiltonian-type functions generate a* $X^\#$ *field. These questions we will begin to answer in the next section.*

2.1.6 Hamiltonian Actions

Definition 2.1.24 (Comoment Map μ_c)**.** *Consider a symplectic manifold* (M, ω)*, and a Lie group action* $G \circlearrowright M$*. Denote the Lie algebra of* G *as* $\mathfrak{g}$*. Then, the comoment map generates fundamental vector fields* $X^\#$ *on* $\mathfrak{X}(M)$*.*

$$\mu_c : \mathfrak{g} \to C^\infty(M) \qquad\qquad X_{\mu_c(X)} = X^\# \tag{2.11}$$

An equivalent condition is that $d\mu_c(X) = \omega(X^\#, \cdot)$*.*

Remark 2.1.15. *The co-moment map* μ_c *contains the information about every Hamilitonian function generating fundamental vector fields on* M*.*

Definition 2.1.25 (Moment Map μ)**.** *There is a natural dual to* $\mu : M \to \mathfrak{g}$

Lemma 2.1.12 (Moment-Comoment Duality)**.** *The comoment and moment maps are connected via the relation:*

$$[\mu_c(X)]\,(p) = \mu^X(p)$$

Definition 2.1.26 (Symplectic Action)**.** *Let* (M, ω) *be a symplectic manifold, with a smooth Lie group action* $G \circlearrowright M$*. A smooth action* $G \circlearrowright M$ *is a symplectic action if for each* $g \in G$*, the map* $g : M \to M$ *is a symplectomorphism. We denote a symplectic action* (M, ω, G)

Definition 2.1.27. *A symplectic action* $G \circlearrowright (M, \omega)$ *is called a Hamiltonian action if there exists a linear* $\mu_c : \mathfrak{g} \to C^\infty(M)$*, the comoment map, such that the diagram commutes:*

$$\begin{tikzcd}
(\mathfrak{g},[\cdot,\cdot]) \arrow[r, "\mu_c"] \arrow[d, "\#"] & (C^\infty(M),\{\cdot,\cdot\}_\omega) \arrow[d, "X."] \\
(\mathcal{S}(M,\omega),[\cdot,\cdot]) & (\mathcal{H}(M,\omega),[\cdot,\cdot]) \arrow[l, hook']
\end{tikzcd}$$

Therefore, a symplectic action is Hamiltonian if its corresponding vector field is itself Hamiltonian, such that the diagram commutes.

Example 2.1.6 (Symplectic Action but not Hamilitonian)**.** *An action that is not Hamiltonian is rotation on the torus,* $S^1 \circlearrowleft T^2$*:*

$$S^1 \circlearrowleft T^2 : \theta \circ (\theta_1, \theta_2) = (\theta_1 + \theta, \theta_2) \tag{2.12}$$

The vector field induced by this action is $\frac{\partial}{\partial \theta_1}$. Taking $\omega = d\theta_1 \wedge d\theta_2$, we have that $\omega(\frac{\partial}{\partial \theta_1}, \cdot) = d\theta_2$. Note $i_{\frac{\partial}{\partial \theta_1}} \omega = d\theta_2$ cannot be everywhere written as the exterior derivative of a Hamiltonian function. Otherwise, we would have for all $z \in S^1$, that $d\theta_2$ is exact, violating the cohomological restriction $H^1_{dR}(S^1) = \mathbb{R}$.

Example 2.1.7 (Rotation on S^2)**.** *An action that is Hamiltonian is rotation on the sphere* S^2*, around the* $\hat{z}$ *axis. Let* $S^2 \subset \mathbb{C} \times \mathbb{R}$*, where* $S^2 = \{(z,h) : |z|^2 + h^2 = 1\}$*. Define the action of* $S^1 \circlearrowleft S^2$ *by:*

$$S^1 \circlearrowleft S^2 : \theta \circ (z, h) = (e^{i\theta} z, h) \tag{2.13}$$

The symplectic form for this action is $d\theta \wedge dh$*, where this is the area form in polar coordinates. The vector field is* $\frac{\partial}{\partial \theta}$*. Therefore, we may declare that* $\mu_c = h$*, and* $d\mu_c = \omega(X_\theta, \cdot)$*.*

Example 2.1.8 (Generalization of Angular Momentum)**.** *The Lie group action* $S^1 \circlearrowleft \mathbb{C}$ *generates the black arrows as a vector field. The action is explicitly:*

$$S^1 \circlearrowleft \mathbb{C} : e^{i\beta} \cdot re^{i\theta} \mapsto re^{i(\theta+\beta)} \tag{2.14}$$

As in differential geometry, vectors are the directional derivatives they induce. Therefore, in polar coordinates the black arrows are the vector field such that:

$$X(e^{i\theta}) = i \cdot e^{i\theta} \tag{2.15}$$

Therefore, $X^\#(\theta, r) = \frac{\partial}{\partial \theta}$*. The symplectic form on* $\mathbb{R}^2$ *is the canonical area form in polar coordinates,* $\omega = rd\theta \wedge dr$*. Now,* $d\mu_c = rdr$*. We find that* $\mu_c = \frac{r^2}{2}$*. Considering a particle moving along the path with mass* $1/2$*, we note that in fact our expression* μ_c *is the classical angular momentum of the particle.*

$$|\mu_c| = L_\theta \tag{2.16}$$

References: Terek (2020), Dundas (2018), Arnold (2010), Cannas da Silva (2001)

2.1.7 Properties of Hamiltonian G-Spaces

An important operation when establishing moment map images is the following:

Definition 2.1.28 (Minkowski Sum). *Given two vector spaces* A, B*, their Minkowski sum is* $A + B = \{a + b : a \in A, b \in B\}$*, where* $+$ *is the vector space addition operator.*

Remark 2.1.16. *A key lemma we will use throughout this thesis is the following: for the product of two manifolds, the moment image of the sum of spaces will be the Minkowski sum of the individual moment images. This is proven below:*

Lemma 2.1.13 (Moment of Product is Minkowski Sum). *Let* $(M_1, \omega_1, G, \mu_1)$ *and* $(M_2, \omega_2, G, \mu_2)$ *be two Hamiltonian G-spaces. It holds that* $(M_1 \times M_2, \omega_1 \oplus \omega_2, G, \mu_1 \oplus \mu_2)$ *is also a Hamiltonian G-space, where the action of* G *on* $M_1 \times M_2$ *is diagonal and the action of the new moment map* $\mu_1 \oplus \mu_2 : M_1 \times M_2 \to \mathfrak{g}^*$ *is given by* $(\mu_1 \oplus \mu_2)(x, y) \doteq \mu_1(x) + \mu_2(y)$*. In particular,* $(\mu_1 \oplus \mu_2)(M_1 \times M_2) = \mu_1(M_1) + \mu_2(M_2)$*, an observation that we will apply later*

Let $i_{x_2}(x) = (x, x_2)$, and $i_{x_1}(x) = (x_1, x)$ be the inclusion maps. Using the fact that $T(M_1 \times M_2) = \pi_1^* TM_1 \oplus \pi_2^* TM_2$, we have that if $f \in C^\infty(M_1 \times M_2)$ then:

$$X_f|_{(x_1,x_2)} = X^{\omega_1}_{f \circ i_{x_2}} + X^{\omega_2}_{f \circ i_{x_1}} \tag{2.17}$$

$$d(\mu_1 \oplus \mu_2)_{(x_1,x_2)}(\cdot) = (\pi_1^*\omega_1 \oplus \pi_2^*\omega_2)(X^{\omega_1}_{f(i_{x_2})} + X^{\omega_2}_{f(i_{x_1})}, \cdot) \tag{2.18}$$

Since opposite projections cancel, i.e., $d\pi_1 d\pi_2 = 0$, we have that the moment map is diagonal, and:

$$(\mu_1 \oplus \mu_2)_{(x_1,x_2)} = \mu_1(x_1) + \mu_2(x_2) \tag{2.19}$$

Example 2.1.9 (Moment Image of $S^2(r_2) \times S^2(r_1)$). *Consider* $(S^2(r_2), \mu_2)$ *where* $\mu_2(r) = r$*, and* $(S^2(r_1), \mu_1)$ *given by* $\mu_2(r) = r$*. This is proven here 8.1.1. Then,* $(\mu_1 \oplus \mu_2)_{(x_1,x_2)} = \mu_1(x_1) + \mu_2(x_2)$*.*

$$(\mu_1 \oplus \mu_2)(S^2(r_2) \times S^2(r_1)) = B(r_2 + r_1) \setminus B(|r_2 - r_1|) \tag{2.20}$$

References: Arnold (2010), Cannas da Silva (2001), Terek (2020)

Chapter 3

Delzant's Theorem

3.1 Delzant's Theorem

In this section we cover the following:

1. What is the statement and proof of Delzant's Theorem
2. Delzant in the $[-1, 1] = \Delta$ case
3. Delzant's Theorem and uniqueness modulo toric equivariant symplectomorphism

3.1.1 Toric Actions and Moment Maps

Toric actions are the most elementary Lie group actions because they are abelian. Let's see the definition of a toric action on complex $\mathbb{C}^d$-space:

Example 3.1.1 (Toric Action $T^n \circlearrowright \mathbb{C}^n$)**.** *One common Lie group action is that of the torus, T^n acting on $\mathbb{C}^n$:*

$$(e^{i\beta_1}, \dots, e^{i\beta_n}) \cdot (e^{i\theta_1}, \dots, e^{i\theta_n}) = (e^{i(\theta_1+\beta_1)}, \dots, e^{i(\theta_n+\beta_n)}) \tag{3.1}$$

The toric action is given by phase rotation in each of the complex coordinates.

A toric action will also have a corresponding vector field. This is important to study as it will be used to find the moment map μ_c

Lemma 3.1.1 (Fundamental Vector Field $X^{\#}$ of Toric Action)**.** *Given a toric action as in 3.1.1,*

$$X^{\#}_{\theta_1,\dots\theta_n} = \frac{\partial}{\partial\theta_1} + \cdots + \frac{\partial}{\partial\theta_n}$$

Proof. We take the derivative and observe:

$$[d\sigma_{\theta_1,\dots,\theta_n}]_e = (ie^{i\theta_1}, \dots, ie^{i\theta_n}) \tag{3.2}$$

But this is the vector field tangent to $e^{i\theta_1}, e^{i\theta_2}, \dots e^{i\theta_n}$. This is, in local coordinates:

$$X^{\#}_{\theta_1,\dots\theta_n} = \frac{\partial}{\partial\theta_1} + \dots + \frac{\partial}{\partial\theta_n}$$

□

Now, we must compute the symplectic form in polar coordinates. This is:

Lemma 3.1.2 (A Symplectic Form on $\mathbb{C}^n$)**.** *We choose a symplectic form:*

$$\omega = \sum_i dz_i \wedge d\overline{z_i} = \sum_i |z_i| d\theta_i \wedge d|z_i|$$

Proof. In 2.1.2, we proved that: $\omega_{\mathbb{C}^k} = \frac{-i}{2}\sum_k dz \wedge \overline{dz}$ was a symplectic form for $(\mathbb{C}^k, \omega_{\mathbb{C}^k})$. We choose a multiple of this form to be our form:

$$\omega = \sum_{j\geq 1} dz_j \wedge d\overline{z_j}$$

Then, we use polar coordinates $dz_j = |z_j| d\theta_j$, and find

$$\omega = \sum_{j\geq 1} |z_j| d\theta_i \wedge d|z_j|$$

□

Now, with a symplectic form ω, and a fundamental vector field, we compute the moment map μ_c on $\mathbb{C}^d$. This is:

Lemma 3.1.3 (Moment Map for $(\mathbb{C}^d, \omega, T^d)$)**.**

$$(\mu(z_1, \dots, z_n))_j = \frac{|z_j|^2}{2} + \lambda_j \tag{3.3}$$

Where $\lambda_j \in \mathbb{R}^*$

Proof. According to $d\mu_c = \omega(X^{\#}, \cdot)$, we observe that: $(d\mu_c)_j = \Big[\omega(X^{\#}_{\theta_1,\dots\theta_n}, \cdot)\Big]_j = |z_j| \cdot d|z_j|$. We integrate both sides to find: $(\mu)_j = \frac{|z_j|^2}{2} + \lambda_j \in (\mathbb{R}^d)^*$ □

References: Guillemin (1994) Terek (2020), Cannas da Silva (2001)

3.1.2 Definitions Concerning the Delzant Construction

Remark 3.1.1. *Now, with basic results behind us for the toric action* $T^d \circlearrowright M$*, we can study the geometry of the image of the moment map as in Delzant (1988).*

Delzant's construction is an association between a symplectic manifold with toric action: $(M_\Delta, \omega_\Delta, T^d)$ to each Delzant space Δ, where Δ is a type of polytope, called a Delzant polytope. Delzant's construction links $M_\Delta \leftrightarrow \Delta$ via the Delzant moment map. Now, we will study the construction:

Remark 3.1.2. *In this case, we denote the Delzant construction moment map:* $J : M_\Delta \to (\mathbb{R}^d)^*$*, where as the moment map* μ_c *denotes the one here* 3.1.3.

Definition 3.1.1 (Delzant Polytope). *A Delzant polytope* (Δ, n, d) *is a convex polytope in* $(\mathbb{R}^n)^*$*. There are d-faces of a Delzant polytope* Δ*, and n edges meeting in each vertex p. The edges meeting at each p are rational. Furthermore,* $v_1, \ldots, v_n$ *can be chosen to be a basis of* $(\mathbb{Z}^n)^*$*. Each one of these faces is defined by a normal equation, where* u_i *is the inward pointing normal vector to each face Delzant (1988)*

$$\langle u_i, x\rangle \geq \lambda_i, \ i \in \{1, \ldots, d\} \tag{3.4}$$

Now, we define several maps used in the Delzant construction:

Lemma 3.1.4 (Delzant Quotient Map). *Let* $e_1, \ldots, e_d$ *be standard basis vectors of* $\mathbb{R}^d$*. There exists a linear map* π'*:*

$$\pi' : T^d \to T^n$$

Proof. Let $e_1, \ldots, e_d$ be standard basis vectors of $\mathbb{R}^d$ and consider the map:

$$\pi : \mathbb{Z}^d \to \mathbb{Z}^n \tag{3.5}$$

Consider its natural extension by inclusion:

$$\pi : \mathbb{R}^d \to \mathbb{R}^n \tag{3.6}$$

Then, there exists a quotient map: $\pi' : T^d \to T^n$, such that the diagram commutes. The quotient map: π' does the following:

$$\pi' : [r] \mapsto [\pi(r)] \tag{3.7}$$

$$\begin{tikzcd}
\mathbb{Z}^d \arrow[r, "\pi"] \arrow[d, hook] & \mathbb{Z}^n \arrow[d, hook] \\
\mathbb{R}^d \arrow[r, "\pi"] \arrow[d, two heads] & \mathbb{R}^n \arrow[d, two heads] \\
T^d \arrow[r, "\pi'"] & T^n
\end{tikzcd}$$

□

Remark 3.1.3. *It is an instructive exercise that π' has a less obvious form. Let the coordinates of each normal vector be $u_i = (u_{i,1}, \ldots, u_{i,n})$. Then,*

$$\pi'(z_1, \ldots, z_d) \mapsto (\prod_{i=1}^{d} z_i^{u_{i,1}}, \ldots, \prod_{i=1}^{d} z_i^{u_{i,n}}) \tag{3.8}$$

Since π' is a linear map, we study its kernel, which we define here:

Definition 3.1.2 (Delzant N Space)**.** *Let the kernel of π' be N.*

Lemma 3.1.5 (Exact Sequence Defining N)**.** $\mathfrak{n}$ *can equivalently be defined by the exact sequence:*

$$0 \to \mathfrak{n} \to \mathbb{R}^d \to \mathbb{R}^n \to 0$$

Proof. This follows from the exact sequence:

$$0 \to N \to T^d \to T^n \to 0$$

Now we take the differential of each map and move to the Lie algebra level:

$$0 \to \mathfrak{n} \to \mathbb{R}^d \to \mathbb{R}^n \to 0$$

The conclusion follows □

References: Cannas da Silva (2001) Guillemin (1994), Terek (2020)

3.1.3 Construction of Delzant Space M_Δ given Δ

Now, for a polytope $\Delta \in (\mathbb{R}^n)^*$, we will construct a symplectic manifold M_Δ, such that its moment map $J : M_\Delta \to \Delta$. First we need a few definitions:

Definition 3.1.3 (Delzant Inclusion Map)**.**

$$i : N \to T^d$$

Definition 3.1.4 (Dual Inclusion Map)**.** *Define:*

$$i^* : (\mathbb{R}^d)^* \to \mathfrak{n}^* \text{ where } i^* \text{ is the derivative transpose of } i \tag{3.9}$$

Lemma 3.1.6 (Almost M_Δ)**.** *The set $(i^* \circ J)^{-1}(0)$ is a compact subset of $\mathbb{C}^d$, and N acts freely on this set.*

Proof. From the exact sequence:

$$0 \to \mathfrak{n} \to \mathbb{R}^d \to \mathbb{R}^n \to 0 \tag{3.10}$$

we conclude that $\ker i^* = \operatorname{Im} \pi^*$. Furthermore, it is an observation that:

$$(i^* \circ \mu_c)^{-1}(0) = \mu_c^{-1}(\pi^*(\Delta)) \tag{3.11}$$

Note that the stabilizer group of z in T^d is T^I, where I is the number of 0's in the coordinate expansion of z. Note when p is in the interior of Δ, the number of 0's for a given z_i is zero. This is because:

$$p \in \operatorname{Int} \Delta \iff \langle u_i, p \rangle > \lambda_i \tag{3.12}$$

When p a vertex of the polygon, assume $p = 0$, so that:

$$\langle u_i, p \rangle = 0, \ i \in \{1, \ldots, n\} \tag{3.13}$$

Then, we note that $\pi(T^I) \cong T^n$ and therefore, we have that $T^I \cap N = \emptyset$. Since T^I is also the stabilizer group, we have that N acts freely at z. Furthermore, note that since J is proper, $(i^* \circ J)^{-1}(0)$ is compact. □

Remark 3.1.4. *In order to define the Delzant Space M_Δ, we need to remove the free action of N on $(i^* \circ \mu_c)^{-1}(0)$. For this, we cite the following Theorem.*

Theorem 3.1.7 (Marsden-Weinstein)**.** *Let (M, ω, G, μ) be a Hamiltonian G-space and $p \in \mathfrak{g}^*$ be a regular value for μ. Assume that the action $G_p \circlearrowright \mu^{-1}(p)$ is free and proper. Then the quotient $\mu^{-1}(p)/G_p$ has a unique symplectic form ω_0 characterized by the relation $\pi^*\omega_0 = i^*\omega$, where $\pi : \mu^{-1}(p) \to \mu^{-1}/G_p$ is the quotient projection and $i : \mu^{-1}(p) \hookrightarrow M$ is the inclusion. We call $(\mu^{-1}(p)/G_p, \omega_0)$ the reduction of (M, ω) to level p. For more details, see Marsden and Weinstein (1974) and Terek (2020)*

The following lemma is important as we reduce:

Lemma 3.1.8 (Moment Map of Lie Subalgebra)**.** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra. Suppose a symplectic manifold (M, ω) is equipped with a Hamiltonian action of a Lie group G with moment map*

$$\mu_c : M \to \mathfrak{g}^*.$$

Let $H \subseteq G$ be a Lie subgroup with Lie algebra $\mathfrak{h}$, and consider the restricted action $H \circlearrowright M$. Then the dual of the inclusion map $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ induces a projection

$$i^* : \mathfrak{g}^* \to \mathfrak{h}^*,$$

and the composition

$$\mu_H := i^* \circ \mu : M \to \mathfrak{h}^*$$

is a moment map for the H-action.

Proof. Since Hamiltonian functions arising from the restricted action $H \circlearrowright M$ are restrictions of Hamiltonian functions arising from $G \circlearrowright M$, it follows that:

$$\mu_{H,c} : \mathfrak{h} \to C^\infty(M) \qquad\qquad \mu_c : \mathfrak{g} \to C^\infty(M) \tag{3.14}$$

are related by $\mu_{H,c} = \mu_c \circ i$. Then, we note that: $\langle \mu, i \circ X \rangle = \mu_{H,c}(X)$. Therefore, $\langle i^* \circ \mu, X \rangle = \mu_{H,c}(X)$, and we note that: $\mu_H = i^* \circ \mu$. ☐

Theorem 3.1.9 (Construction of M_Δ). *Given $\Delta \in (\mathbb{R}^n)^*$ a Delzant Polytope, there exists: $(M_\Delta, \omega_0, T^n)$ such that $J(M_\Delta) = \Delta$*

Proof. From the Marsden-Weinstein Theorem 3.1.7, we reduce $\mathbb{C}^d$ with respect to N. Since N acts freely on $(i^* \circ J)^{-1}(0)$, the reduced space is a smooth symplectic manifold, which we declare M_Δ. The symplectic manifold M_Δ admits a T^n action after reduction. Define μ_c to be the moment map μ_c arising from the action of T^n on M_Δ. Consider the inclusion $i' : \mathbb{R}^n \to \mathbb{R}^d$ and its transpose, $(i')^* : (\mathbb{R}^d)^* \to (\mathbb{R}^n)^*$. By 3.1.8, the moment image of M_Δ is by 3.11, $J(M_\Delta) = \Delta$

$$J = (i')^* \circ \mu_c \qquad M_\Delta = (i^* \circ \mu_c)^{-1}(0)/N \qquad J(M_\Delta) \cong \Delta \in \mathbb{R}^{n*} \tag{3.15}$$

☐

References: Cannas da Silva (2001) Guillemin (1994), Terek (2020)

3.1.4 Delzant Construction for $\Delta = [-1, 1]$

We will do the Delzant Construction for $\Delta = [-1, 1]$.

Question (What is N given Δ). *Claim: $N = \{(t,t) : t \in [0, 2\pi)\}$.*

Proof. When $\Delta = [-1, 1]$, the normal vectors are $u_1 = -1$ and $u_2 = 1$. In this case, $N = \ker \pi'$ where:

$$\pi'(z_1, z_2) = z_1^{-1} z_2 \tag{3.16}$$

We use Remark 3.1.3, and observe that $N = \{(t,t) : t \in [0, 2\pi)\}$. ☐

Question (What is μ given Δ). *Claim: $J(z) = \frac{(|z_1|^2, |z_2|^2)}{2} + (-1, -1)$*

Proof. Since the interval Δ can also be seen as $x \geq -1$ and $-x \geq -1$ we have that $\lambda_1 = -1$ and $\lambda_2 = -1$. Using 2.1.2, the complex space $\mathbb{C}^2$ has moment map: $\mu(z) = \frac{(|z_1|^2, |z_2|^2)}{2} + (-1, -1)$ ☐

Question (What is J given Δ). *Claim: $J = i^* \circ J = \frac{|z_1|^2 + |z_2|^2}{2} - 2$*

Proof. Note that $i : \mathfrak{n} \to \mathbb{R}^2$, given by $v \mapsto (v, v)$. Note that $\langle i^*(z_1, z_2), v\rangle = \langle (z_1, z_2), (v, v)\rangle = (z_1 + z_2)v$. We use the property that $\langle a^* w, v\rangle = \langle w, av\rangle$. Therefore, $i^*(z_1, z_2) = z_1 + z_2$, and we note that:

$$i^* \circ J = \frac{|z_1|^2 + |z_2|^2}{2} - 2$$

□

Question (What is M_Δ). *Claim:* $M_\Delta \cong \mathbb{C}P^1$

Proof. We note that $N \cong U(1)$. Therefore, $X_\Delta \cong S^3/N \cong S^3/U(1) \cong \mathbb{CP}^1$. □

3.1.5 Convexity of Moment Maps

For the other direction, we use the Atiyah, Guillemin, and Sternberg (1982). Following the treatment in Cannas da Silva (2001), we sketch only one part of the proof.

Theorem 3.1.10 (Atiyah-Guillemin-Sternberg). *Let (M, ω) be a compact connected symplectic manifold, and let T^m be an m-torus. Suppose that $\psi : T^m \to \mathrm{Symp}(M, \omega)$ is a Hamiltonian action with moment map $\mu : M \to \mathbb{R}^m$. Then:*

1. *the levels of μ are connected;*
2. *the image of μ is convex;*
3. *the image of μ is the convex hull of the images of the fixed points of the action.*

Consider the statements:

1. A_n - the levels of μ are connected for any T^n action
2. B_n - the image of μ is convex for any T^n action

From Morse Theory, we have that A_1 and B_1 hold.

Theorem 3.1.11 (Part of Proof of AGS). *A_{m-1} implies B_m*

Proof. Consider a moment map μ from a toric action $T^n \circlearrowright M$. We can consider the toric action of an $(n-1)$-dimensional sub-torus on M, $T^{n-1} \circlearrowright M$, by using an injective matrix $A \in \mathbb{Z}^{n\times(n-1)}$ and defining:

$$\psi_A : T^{n-1} \to \mathrm{Symp}(M, \omega) \qquad\qquad \theta \mapsto \psi_{A\theta} \tag{3.17}$$

Since $A^T : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is a linear map from the Lie algebra of the torus onto the Lie algebra of its subgroup, the moment map of ψ_A is $\mu_A = A^T\mu$. Consider $p, p_0 \in \mu_A^{-1}(\xi)$, then $A^T\mu(p) = \xi = A^T\mu(p_0)$. Now it follows that:

$$\mu_A^{-1}(\xi) = \{p \in M : \mu(p) - \mu(p_0) \in \mathrm{Ker}(A^T)\} \tag{3.18}$$

Since $\mu_A^{-1}(\xi)$ is a connected manifold for regular value ξ and is locally path connected, the inverse image is path connected. Therefore, for a path $t \mapsto p_t \in \mu_A^{-1}(\xi)$, there also exists a path $t \mapsto \mu(p_t) - \mu(p_0) \in \mathrm{Ker}(A^T)$. Since $\mathrm{Ker}(A^T)$ is 1-dimensional, it follows that all elements $\mu(p_t)$ must be a linear superposition of $\mu(p_0)$ and $\mu(p_1)$ - two points determine a line!

$$(1-t)\mu(p_0) + t\mu(p_1) \in \mu(M),\ t \in [0,1] \tag{3.19}$$

Choosing points p_0 and p_1 closely approximated by rational p_0' and p_1' we run this argument again - finding a matrix which contains $\mu(p_0') - \mu(p_1')$ in its kernel. Taking limits, we find that $\mu(M)$ is convex. □

References: Cannas da Silva (2001), Delzant (1988), Terek (2020)

3.1.6 Uniqueness of Delzant Spaces

Now, we have shown that for each Delzant polytope, there exists a corresponding Delzant space, and given a Delzant space, we may find the polytope. Actually, the association of Δ and M_Δ is unique, up to toric-equivariant symplectomorphism as in Delzant (1988).

Definition 3.1.5. *Given a Delzant polytope Δ, if there exist moment maps μ_1 and μ_2, and Delzant spaces, $X_{\Delta,1}$ and $X_{\Delta,2}$, such that there exists an isomorphism $\gamma : \mathrm{Im}(\mu_1) \to \mathrm{Im}(\mu_2)$ and $\phi : X_{\Delta,1} \to X_{\Delta,2}$ is a γ-equivariant symplectomorphism, then ϕ is called a γ-equivariant symplectomorphism, which is a diffeomorphism in the symplectic category that respects the moment maps.*

$$\begin{tikzcd}
X_{\Delta,1} \arrow[r, "\phi"] \arrow[d, "\mu_1"'] & X_{\Delta,2} \arrow[d, "\mu_2"] \\
\mathrm{Im}(\mu_1) \arrow[r, "\gamma"'] & \mathrm{Im}(\mu_2)
\end{tikzcd}$$

Now, we would like to extend these results to the non-Abelian case $G \circlearrowright M$. It turns out that so long as the action of G has regular values, the uniqueness of the moment map image is determined by the toric (Abelian) part of G, which is known as the maximal torus of G.
References: Blaom (1996), Guillemin and Sternberg (1982).

Chapter 4

Moment Maps for SO(3)-actions

4.1 Moment Maps of Lie Actions Arising from SO(3)

Now, we would like to begin developing a generalization of Delzant's Theorem for a symplectic manifolds admitting a Hamilitonian SO(3) action, $(M, \omega, \mathrm{SO}(3))$.

Remark 4.1.1. *As a result, the moment map for such an action will be* $\mu : M \to \mathfrak{so}(3)^*$. *As a result, we must study the Lie algebra* $\mathfrak{so}(3)^*$

In this section we cover the following:

1. Coadjoint action SO(3) is given by evaluation
2. The Lie-algebra isomorphism $\mathfrak{so}^*(3) \cong \mathbb{R}^3$
3. The classification of moment images of symplectic manifold with Hamilitonian SO(3) action, $(M, \omega, \mathrm{SO}(3))$.

4.1.1 Coadjoint Action

To classify moment map images, we must first show that the natural action of SO(3) on $\mathfrak{so}(3)$ is itself given by evaluation.

Definition 4.1.1 (Coadjoint Action)**.** *Given a Lie group* G*, and its Lie algebra* $\mathfrak{g}$*, there is a natural action of* $G \circlearrowright \mathfrak{g}$ *given by the coadjoint action. For each covector* $h \in \mathfrak{g}^*$, $A \cdot h \to \mathrm{Ad}^*_{A^{-1}} h$

Lemma 4.1.1. *The map* $\Phi : \mathfrak{so}(3)^* \to \mathbb{R}^3$ *defined by*

$$\Phi(\xi) = (\xi(E_1), \xi(E_2), \xi(E_3))$$

is an SO(3)*-equivariant vector space isomorphism.*

Proof. Since $\dim \mathfrak{so}(3) = \dim \mathfrak{so}(3)^* = 3$, it suffices to show that Φ is injective. Suppose $\Phi(\xi) = 0$. Then ξ vanishes on the basis $\{E_1, E_2, E_3\}$ of $\mathfrak{so}(3)$, so $\xi = 0$. Hence, $\ker \Phi = 0$, and Φ is injective. ☐

Remark 4.1.2. *Now, we would like to prove that the coadjoint action on* $\mathfrak{so}(3)^*$ *is given by evaluation.*

Lemma 4.1.2 (Coadjoint Action is Given by Evaluation)**.** *Let* $\xi \in \mathfrak{so}(3)^*$*, and* Φ *the evaluation map as above. Then:*

$$\Phi\left(\xi \circ \mathrm{Ad}_{A^{-1}}\right) = A\Phi(\xi)$$

Proof. Under the isomorphism Φ, and noting that $A^{-1} = A^T$ for $A \in \mathrm{SO}(3)$, we want to show $A\Phi(\xi) = \big(\xi(A^T E_1 A),\ \xi(A^T E_2 A),\ \xi(A^T E_3 A)\big)$. Let $\xi(E_1) = \xi_1$, $\xi(E_2) = \xi_2$, and $\xi(E_3) = \xi_3$. It suffices to verify (4.1.1) on the basis elements. We verify this for the first basis element, the other cases follow by a similar computation.

$$\xi\left(A^T E_1 A\right) = \xi\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T E_1 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) \tag{4.1}$$

$$= \xi\left(\begin{bmatrix} 0 & -a_{21}a_{32} + a_{31}a_{22} & -a_{21}a_{33} + a_{31}a_{23} \\ -a_{22}a_{31} + a_{21}a_{32} & 0 & -a_{22}a_{33} + a_{32}a_{23} \\ -a_{23}a_{31} + a_{21}a_{33} & -a_{23}a_{32} + a_{22}a_{33} & 0 \end{bmatrix}\right) \tag{4.2}$$

We compute this as a linear combination of the basis matrices:

$$\xi(A^T E_1 A) = \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \xi(E_1) + \det\begin{bmatrix} a_{22} & a_{32} \\ a_{21} & a_{31} \end{bmatrix} \xi(E_2) + \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \xi(E_3) \tag{4.3}$$

$$= \det\begin{bmatrix} \xi_1(E_1) & \xi_2(E_2 & \xi_3(E_3) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{4.4}$$

$$= \left\langle \Phi(\xi),\ A^T e_2 \times A^T e_3 \right\rangle = \langle A\xi,\ e_1 \rangle = [A\Phi(\xi)]_1 \tag{4.5}$$

The same holds for components 2 and 3. Therefore,

$$\Phi\left(\mathrm{Ad}^* A \cdot \xi\right)_j = [A\Phi(\xi)]_j \quad \text{for } j = 1, 2, 3$$

☐

References: Blaom (1996)

4.1.2 Lie Algebra of $\mathfrak{so}(3)$

Now, we have found the the coadjoint action $\mathrm{Ad}^*_{A^{-1}}$, is given by evalulation. Now, we would like to develop the Lie algebra isomorphism between $(\mathbb{R}^3, \times)$ and $(\mathfrak{so}(3), [\cdot, \cdot])$. This isomorphism will allow us to interpert the images of moment maps $\mu : M \to \mathfrak{so}(3)^*$ as though they were in Euclidean space.

Remark 4.1.3 (Lie algebra of $\mathfrak{so}^3$)**.** *The Lie algebra of* $\mathfrak{so}^3$ *has the Lie bracket given by the matrix commutator. Therefore, we denote the Lie algebra as* $(\mathfrak{so}_3$*. Furthermore, The Lie algebra of* $\mathbb{R}^3$ *has the Lie bracket given by the cross product. Therefore, we denote the Lie algebra as* $(\mathbb{R}^3, \times)$*.*

Lemma 4.1.3 (Lie Algebra Isomorphism)**.** $(\mathbb{R}^3, \times) \to (\mathfrak{so}(3), [\cdot, \cdot])$ *given by*

$$\phi : (x_1, x_2, x_3) \to \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{so}(3) \tag{4.6}$$

is an isomorphism of Lie algebras

Proof. Note that ϕ is injective. Furthermore, we note that $\dim \mathfrak{so}(3) = \mathbb{R}^3$. Therefore, we know that ϕ is an isomorphism of vector spaces. Now, we show that Lie bracket is preserved under isomorphism. Let $X = (x_1, x_2, x_3)$, and $Y = (y_1, y_2, y_3)$. Then, we note that:

$$[X, Y] = X \times Y = (x_2y_3 - x_3y_2, x_3y_1 - y_3x_1, x_1y_2 - y_1x_2)$$

Therefore, we have that

$$\phi([X, Y]) = \begin{bmatrix} 0 & -x_1y_2 + y_1x_2 & x_3y_1 - y_3x_1 \\ x_1y_2 - y_1x_2 & 0 & -x_2y_3 + x_3y_2 \\ -x_3y_1 + y_3x_1 & x_2y_3 - x_3y_2 & 0 \end{bmatrix}$$

Alternatively, when we compute

$$[\phi(X), \phi(Y)] = \phi(X)\phi(Y) - \phi(Y)\phi(X)$$

We notice that the following occurs:

$$[\phi(X), \phi(Y)] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -x_1y_2 + y_1x_2 & x_3y_1 - y_3x_1 \\ x_1y_2 - y_1x_2 & 0 & -x_2y_3 + x_3y_2 \\ -x_3y_1 + y_3x_1 & x_2y_3 - x_3y_2 & 0 \end{bmatrix}$$

Therefore, we conclude. □

Remark 4.1.4. *As a consequence, this will establish that there is a Lie algebra isomorphism between* $(\mathfrak{so}(3)^*, [\cdot,\cdot])$ *and* $(\mathbb{R}^3, \times)$ *as well.*

Remark 4.1.5. *Now, there exists a Lie algebra isomorphism between* $\mathfrak{so}(3)^*$ *and* $\mathbb{R}^3$ *by chasing the diagram:*

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "\phi"] & \mathfrak{so}(3) \arrow[d] \\
\mathbb{R}^3 \arrow[u] & \mathfrak{so}(3)^* \arrow[l, "\Phi"']
\end{tikzcd}$$

Remark 4.1.6. *Now, we study the images of* μ *moment maps arising from a Hamilitonian* SO(3)*-action on* $(M, \omega, \mathrm{SO}(3))$*. We use the above results freely 4.1.2, and 4.1.3*

4.1.3 Images of Moment Maps

Now, we define a specific notion of equivariance

Definition 4.1.2 (G-Equivariance)**.** *Let* G *be a Lie group action both on* $G \circlearrowright X$ *and on* $G \circlearrowright Y$*. Let* $f : X \to Y$ *a diffeomorphism. Then* f *is* G*-equivariant if:*

$$f(g \cdot x) = \phi(g) \cdot f(x)$$

Consider a symplectic manifold $(M, \omega, \mathrm{SO}(3))$, with moment map $\mu : M \to \mathfrak{so}_3^*$. We show a useful lemma, that is coadjoint action acts equivariantly on the moment map. This will allow us to control the moment map image.

Lemma 4.1.4 (Coadjoint Action is A-Equivariant)**.** *Consider a connected symplectic manifold* $(M, \omega, \mathrm{SO}(3))$*, and* $A \in \mathrm{SO}(3)$*. Then, we have that:*

$$\mu(A \cdot x) = Ad^*_{A^{-1}}\mu(x)$$

where $Ad^*_{A^{-1}}$ *denotes the pullback.*

Proof. It is enough to prove the differential identity on both sides. Namely, observe that the above statement is equivalent to:

$$d\mu_x(X^\#_M|_x) = \mu(x) \circ \mathrm{ad}(-X, \cdot)$$

Now, consider $Y \in \mathfrak{so}(3)$. Then, we note that

$$\mu_x(X^\#_M|_x)Y = \omega_x(X^\#_M|_x, Y^\#_M|_x) = \{\mu^X, \mu^Y\} = \mu^{[X,Y]}(x) = \mu(x)[Y, X] \tag{4.7}$$

Now, we manipulate the expression $\mu(x) \circ \mathrm{ad}(-X, \cdot)$, and find:

$$[\mu(x) \circ ad(-X, \circ)]\, Y = \mu(x)[-X, Y] = \mu(x)[Y, X] \tag{4.8}$$

Since both expressions agree on the differential level, the conclusion follows. □

Now, since the coadjoint action is A-Equivariant, and the coadjoint action is given by evaluations, we have that μ is $\mathrm{SO}(3)$ invariant. Now, we use that $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ to invoke that images of moment maps $\mu : M \to \mathbb{R}^3$ must have $\mathrm{SO}(3)$ symmetry.

References: Terek (2020)

4.1.4 Sets that are $\mathrm{SO}(n)$ Invariant

Lemma 4.1.5. *Let* $\mathrm{SO}(n)$ *act on* $\mathbb{R}^n$ *by the standard action, and let* $K \subset \mathbb{R}^n$ *be* $\mathrm{SO}(n)$*-invariant. We define* $\|K\| = \{\|w\| : w \in K\}$*. Then:*

$$K = \bigcup_{r \in \|K\|} S^{n-1}(r) \tag{4.9}$$

If K *is connected, in particular, one of the following occurs*

1. $K = \emptyset$
2. $K = S^{n-1}(r)$ *for some* $r > 0$
3. $K = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ *for some* $r > 0$
4. $K = \{x \in \mathbb{R}^n : r_1 \leq \|x\| \leq r_2\}$ *for* $0 < r_1 < r_2$

Proof. We observe that for each $p \in \mathbb{R}^n$:

$$\mathrm{SO}(n) \cdot p = \{y \in \mathbb{R}^n : \|y\| = \|p\|\} \tag{4.10}$$

Therefore, as $p \in K$, it holds that $\mathrm{SO}(n) \cdot p \subset K$. So:

$$\bigcup_{r \in \|K\|} S^{n-1}(r) = \bigcup_{p \in K} \mathrm{SO}(n) \cdot p \subset K \tag{4.11}$$

The reverse inclusion follows from the observation that if $w \in K$, then $\|w\| \in \|K\|$ and $w \in S^{n-1}(\|w\|)$. Now, by continuity of the norm function $x \mapsto \|x\|$, if K is compact and connected, so will be $\|K\|$. Therefore, $\|K\| = [r_1, r_2]$ where $0 \leq r_1 \leq r_2 < \infty$. □

Theorem 4.1.6. *Let* $r_1, r_2 \in [0, \infty)$ *such that* $r_1 < r_2$*. Then:*

$$S^{n-1}(r_1) + S^{n-1}(r_2) = \bigcup_{r \in [r_2 - r_1, r_2 + r_1]} S^{n-1}(r) \tag{4.12}$$

where the sum on the left is the Minkowski sum of the two spheres

Proof. If $y \in S^k(r_1) + S^k(r_2)$ then $y = x_1 + x_2$ where $x_1 \in S^k(r_1)$ and $x_2 \in S^k(r_2)$. By the triangle inequality:

$$r_2 - r_1 = \|x_2\| - \|x_1\| \leq \|y\| \leq \|x_2\| + \|x_1\| = r_2 + r_1 \tag{4.13}$$

Therefore,

$$y \in \bigcup_{r \in [r_2 - r_1, r_2 + r_1]} S^k(r) \tag{4.14}$$

For the other inclusion, by $\mathrm{SO}(k+1)$ invariance, we conclude without loss of generality that $k = 1$. By SO(2) invariance, we further assume that $y = (0, r)$ with $r \in [r_2 - r_1, r_2 + r_1]$. If:

$$x_1 = (r_1 \sin\theta_1, r_1 \cos\theta_1) \tag{4.15}$$

$$x_2 = (r_2 \sin\theta_2, r_2 \cos\theta_2) \tag{4.16}$$

The condition $r \in [r_2 - r_1, r_2 + r_1]$ means that there are θ_1 and θ_2 such that:

$$r_2 \sin\theta_2 + r_1 \sin\theta_1 = 0 \tag{4.17}$$

$$r_2 \cos\theta_2 + r_1 \cos\theta_1 = r \tag{4.18}$$

□

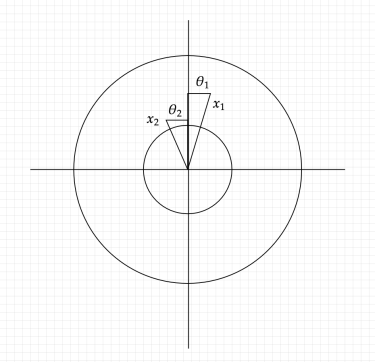


Figure 4.1: Geometric visualization of the Minkowski sum of two spheres $S^{n-1}(r_1)$ and $S^{n-1}(r_2)$, showing how the resulting set forms a spherical shell with inner radius $r_2 - r_1$ and outer radius $r_2 + r_1$.

This geometric interpretation of the Minkowski sum of spheres is fundamental for understanding the structure of moment-image sets for Hamiltonian SO(3)-spaces. Now, we can classify all possible moment-images of compact connected Hamiltonian SO(3)-spaces as being one of four types: a point (the origin), a sphere, a solid ball, or a spherical shell.

References: Terek (2020)

Chapter 5

Structure of SO(3)

5.1 Generalizing Delzant to the SO(3) Case

Now, let's begin outlining a Delzant correspondence, where the Lie group acting on our manifold is no longer T^n, but rather $G = \mathrm{SO}(3)$. The key to our approach will be developing several tools:

1. Understanding SO(3) as a composition of shells and rays as in Blaom (1996).
2. Ultimately, these two points will allow us to build the bundle: $\pi_{\mathcal{O}} : P \to \mathcal{O}$. This allows us to study how a Delzant-type correspondence relates abelian toric actions and nonabelian rotation SO(3) actions.

First, we would like to establish that S^1 is the maximum torus of SO(3).

5.1.1 Maximal Tori of SO(3)

Lemma 5.1.1 (No 2-Dimensional Subalgebras). *There is no 2-dimensional Lie-subalgebra of* $\mathbb{R}^3$

Proof. If $\mathfrak{h}$ is a 2-dimensional subalgebra of $(\mathbb{R}^3, \times)$ and $\{v, w\}$ is a basis for $\mathfrak{h}$ then $v \times w \in \mathfrak{h} \cap \mathfrak{h}^{\perp} = \{0\}$, so $v \times w = 0$ and $\{v, w\}$ is linearly dependent, a contradiction. □

Now, this will be used to prove that:

Lemma 5.1.2 (Maximal Torus of SO(3)). *S^1 is tha maximal torus of* SO(3)

Proof. We note that $(\mathfrak{so}(3), [\cdot, \cdot]) \cong (\mathbb{R}^3, \times)$, and that S^1 is a tori of SO(3).Suppose there existed a torus T inside SO(3). Then, we find that on the Lie-algebra level, $\mathfrak{t} \subset \mathbb{R}^2$ is a 2-dimensional subalgebra. Contradiction, so S^1 is the maximum torus of SO(3). □

References: Wang (2013)

5.1.2 Weyl Chambers and Coadjoint Orbits

Now, we would like to decompose $\mathfrak{so}(3)$ using spherical coordinates

Theorem 5.1.3. *For each point* $\mu_0 \in \mathfrak{g}^*_{reg}$*, there passes a unique Weyl chamber. Furthermore, the intersection of an orbit and Weyl chamber is transverse.*

Proof. Equipped with the isomorphism, for each point $p \in (\mathbb{R}^3) \setminus \{0\}$, take the line in the dual space passing through p. Therefore, Weyl chambers are rays from the origin. The coadjoint orbits are spheres, and it suffices to note that the intersection of a ray and a sphere is transverse. □

Theorem 5.1.4. *If* $\mathcal{W}$ *is a Weyl chamber in* $\mathfrak{g}^*$*, then* $G_{w_1} = G_{w_2}$*, for* $w_1, w_2 \in \mathcal{W}$

Proof. Since w_1, w_2 are in the same Weyl chamber, one is a positive scalar multiple of the other. Then, taking exponentials, their stabilizer subgroups are the same. □

Theorem 5.1.5. *The following is a diffeomorphism:*

$$\gamma : \mathbb{R}^3 \setminus \{0\} \to (0, \infty) \times S^2 \tag{5.1}$$

$$\gamma(x) = \left(\|x\|, \frac{x}{\|x\|} \right) \tag{5.2}$$

Proof. The diffeomorphism is spherical coordinates taking on $\mathbb{R}^3$. To each point we identify a direction and a radius. Therefore, there exists a diffeomorphism between the product of a Weyl chamber and an associated orbit. □

References: Blaom (1996)

5.1.3 Moment Maps and Fibre Bundles

Theorem 5.1.6. *If* $(P, \omega, \mathrm{SO}(3), J)$ *is a Hamiltonian* SO(3)*-space with regular momenta, then* $\dim \mathrm{SO}(3)_x \leq 1$ *for each* $x \in P$

Proof. By SO(3)-equivariance of J, we know that $dJ_x(X^{\#}_p(x)) = X^{\#}_{\mathfrak{so}(3)^{\#}}(J(x))$ for all $x \in P$. If $X \in \mathfrak{so}(3)_x$, then $X^{\#}_P(x) = 0$, and so:

$$X^{\#}_{\mathfrak{so}(3)^{\#}}(J(x)) = 0 \in \mathfrak{so}(3)_{J(x)} \tag{5.3}$$

Now, $\mathfrak{so}(3)_{J(x)}$ cannot have dimension 3 by the regular momentum assumption, and since $\mathbb{R}^3$ does not admit two-dimensional subalgebras, it has dimension less than or equal to 1:

$$\mathfrak{so}(3)_x \subset \mathfrak{so}(3)_{J(x)} \Rightarrow \dim \mathfrak{so}(3)_x \leq 1 \tag{5.4}$$

□

Define the two key projections:

$$\pi_{\mathcal{O}} : P \to S^2,\ \pi_{\mathcal{O}}(x) = \frac{J(x)}{\|J(x)\|} \qquad \pi_{\mathcal{W}} : P \to \mathbb{R}^+,\ \pi_{\mathcal{W}}(x) = \|J(x)\| \tag{5.5}$$

Lemma 5.1.7. *$\pi = \pi_{\mathcal{O}}$ is surjective*

Proof. Since the momentum image is not trivial, there exists elements $g \in G$, $w \in \mathcal{W}$, and $x \in P$ such that $J(x) = g \cdot w$, where J is the momentum map. This implies that $G \cdot w \in \operatorname{Im}(J)$. Now, we note that $\pi_{\mathcal{O}}$ maps each coadjoint orbit diffeomorphically onto $\mathcal{O}$. Therefore: there exists $h \in G \cdot w$ such that $\pi_{\mathcal{O}}(h) = \mu$. Furthermore, there exists $x' \in P$ such that $J(x') = \nu$. This proves that $\pi_{\mathcal{O}} \circ J$ is surjective. □

Theorem 5.1.8. *$\pi^{-1}(\mu)$ is a fibre bundle $P \to \mathcal{O}$:*

Proof. We can verify that the differential map $d\pi_{\mathcal{O}}|_{J(x)} \circ dJ_x$ is surjective. Therefore, each fibre $\pi^{-1}(\mu)$ forms a submanifold of P. Now, we must construct local trivializations to complete the proof. Consider a point $\mu_0 \in \mathcal{O}$ and its isotropy subgroup G_{μ_0}. For any neighborhood U of μ_0, we construct a G_{μ_0}-invariant neighborhood:

$$U' = \bigcup_{h \in G_{\mu_0}} h(U) \tag{5.6}$$

Since the actions of G by left multiplication and its inverse are continuous, the following sets are open:

$$U'_1 = \bigcup_{g \in G} g \cdot U' \tag{5.7}$$

$$U'_2 = \bigcup_{g \in G} g^{-1} \cdot U' \tag{5.8}$$

Taking the intersection $U = U'_1 \cap U'_2$, we obtain a neighborhood with two important properties:

1. U is G_{μ_0}-invariant
2. If $g \cdot \mu_0 \in U$, then $g^{-1} \cdot \mu_0 \in U$

Now, choose s to be a local section of the principal G_{μ_0}-bundle $G \to G/G_{\mu_0}$ such that:

$$s(gG_{\mu_0}) = g \tag{5.9}$$

Also, define the diffeomorphism:

$$\phi_{\mu_0}(gG_{\mu_0}) = g \cdot \mu_0 \tag{5.10}$$

The map $\psi : U \times F \to \pi^{-1}(U)$ defined by:

$$\psi(\mu, x) = s(\phi_{\mu_0}^{-1}(\mu)) \cdot x \tag{5.11}$$

is a local bundle chart, where F denotes the fibre. First, we verify that ψ maps correctly by examining its composition with π:

$$\pi(\psi(\mu, x)) = \pi(s(\phi_{\mu_0}^{-1}(\mu)) \cdot x) = \phi_{\mu_0}(\phi_{\mu_0}^{-1}(\mu)) = \mu \tag{5.12}$$

This shows that ψ preserves the fibres of π. Next, we show that ψ is well-defined and invertible. For any $y \in \pi^{-1}(U)$:

$$\psi(\psi^{-1}(y)) = s \circ \phi_{\mu_0}^{-1} \circ \pi \circ \phi_{\mu_0} \circ s^{-1}(y) = y \tag{5.13}$$

Since both ψ and ψ^{-1} are smooth maps, we conclude that ψ is a diffeomorphism and thus a local bundle chart. The inverse map can be expressed as:

$$\psi^{-1}(y) = (\pi(y), (s(\phi_{\mu_0}^{-1}(\pi(y))))^{-1} \cdot y) \tag{5.14}$$

The local charts for our bundle are:

$$U_\mu = \{g \cdot U : g \in \phi_{\mu_0}^{-1}(\mu)\} \tag{5.15}$$

Recall that $\phi_{\mu_0}^{-1}(\mu) = gG_{\mu_0}$ corresponds to elements satisfying $g \cdot \mu_0 = \mu$. Since $\mathcal{O}$ is compact, we can extract a finite subcover $\{U_{\mu_k}\}_{k=1}^{n}$ of these sets. For each k, we define the local trivialization:

$$\psi_k : U_{\mu_k} \times F \to \pi^{-1}(U_{\mu_k}) \tag{5.16}$$

$$\psi_k(\mu, x) = g_k \cdot \psi(g_k^{-1} \cdot \mu, x) \tag{5.17}$$

where $g_k \in G$ satisfies $g_k \cdot \mu_0 = \mu_k$. The transition functions between overlapping charts are:

$$\psi_j^{-1} \circ \psi_k : (U_{\mu_j} \cap U_{\mu_k}) \times F \to (U_{\mu_j} \cap U_{\mu_k}) \times F \tag{5.18}$$

$$\psi_j^{-1} \circ \psi_k(\mu, x) = (\mu, g_j^{-1} g_k \cdot x) \tag{5.19}$$

These transition functions take values in a subgroup of the maximal torus T, which completes the proof that $\pi : P \to \mathcal{O}$ is a fibre bundle with structure group contained in T. □

Theorem 5.1.9 (Blaom (1996)). *$\pi : P \to \mathcal{O}$ is a fibre bundle, with the structure group of the bundle being a subgroup of the maximal torus, T.*

5.1.4 Symplectic Properties of Moment Maps

Theorem 5.1.10 (Guillemin and Sternberg). *Let Z be a submanifold of $\mathfrak{g}^*$ such that J is transverse to Z and: $T_zZ \oplus T_z(G \cdot z) = \mathfrak{g}^*$. Then $J^{-1}(Z)$ is a symplectic submanifold of P.*

Lemma 5.1.11. *By equivariance, dJ_x spans the tangent space to a coadjoint orbit $\mathcal{O}$. Since $Z = \mathcal{W}$ is transverse to $\mathcal{O}$, it follows that dJ_x intersects $\mathcal{W}$ transversely, and we can apply Theorem 5.1.10 to $Z = \mathcal{W}$, to find that: $J^{-1}(\mathcal{W})$ is a symplectic submanifold. Note that fibres of $\pi : P \to \mathcal{O}$ are just $J^{-1}(\mathcal{W})$. Therefore, using the symplectic form on each submanifold, we have a smoothly varying choice of vector spaces $V_x = \operatorname{Ker} d\pi_x$: $T_xP \cong V_x \oplus V_x^\omega$ This can be phrased as there exists a projection map: $\beta : TP \to \ker d\pi_x$*

Theorem 5.1.12 (Blaom 1996)**.** *Let $v \in \operatorname{Ker} dJ_x$. Then $v \in [T_xJ(G \cdot x)]^\omega$.*

Proof. The result follows from the identity

$$dJ_x(w) = \omega(\xi, w), \quad \xi \in T_xJ(G \cdot x). \tag{5.20}$$

□

Theorem 5.1.13. *The horizontal subspace satisfies*

$$\mathrm{Hor}_x \subset T_x(G \cdot x). \tag{5.21}$$

Proof. Recall that $\mathrm{Hor}_x = \ker d\pi_x$. Since $\operatorname{Ker} dJ_x \subset \operatorname{Ker} d\pi_x$, we have $\operatorname{Ker} dJ_x \subset \mathrm{Hor}_x$. By Theorem 5.1.12, we obtain $\mathrm{Hor}_x \subset T_x(G \cdot x)$. □

Theorem 5.1.14. *The fundamental vector field map $\xi \mapsto \xi_P$ maps $\mathfrak{t}^-$ onto Hor_x.*

Proof. By Theorem 5.1.13, it suffices to establish injectivity, i.e., $\mathfrak{g}_x \cap \mathfrak{t}^- = \{0\}$. This follows immediately from the fact that $\mathfrak{g}_x = \mathfrak{t}$. □

Theorem 5.1.15. *The differential of the momentum map dJ_x sends*

$$\mathrm{Hor}_x \ \textit{ to } T_{J(x)}(G \cdot J(x)) \cong \underline{\mathfrak{t}}^- \subset \mathfrak{g}^* \tag{5.22}$$

Proof. We need to show that $\mathrm{Hor}_x \cap \operatorname{Ker} dJ_x = \{0\}$. Since

$$\operatorname{Ker} dJ_x \subset \operatorname{Ker} d\pi_x \qquad\qquad \mathrm{Hor}_x \cap \mathrm{Ver}_x = \{0\} \tag{5.23}$$

the result follows. □

Theorem 5.1.16. *For any $\xi, \nu \in \mathfrak{g}$, the symplectic form satisfies*

$$\omega(\xi_P(x), \nu_P(x)) = \langle J(x), [\xi, \nu] \rangle, \tag{5.24}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Proof. The following computation yields the necessary result:

$$\omega(\xi_P(x), \nu_P(x)) = \omega(X_{J_\xi}(x), X_{J_\nu}(x)) = J_{[\xi,\nu]}(x) = \langle J(x), [\xi, \nu] \rangle. \tag{5.25}$$

□

Remark 5.1.1. *Theorem 5.1.16 establishes the fundamental relationship between the symplectic structure on P and the Lie algebra structure of $\mathfrak{so}(3)$, completing the generalization of Delzant's correspondence to the* SO(3) *case described in Blaom (1996)*

References: Blaom (1996)

Chapter 6

T-Actions and Further Generalizations

6.1 Canonical T-Action and Generalization of Delzant's Theorem

In this last section, we were studying the fibre $\pi : P \to \mathcal{O}$, and we noted that the transition functions between fibres were contained in a fundamental torus G_μ. Now, we want to use this fact to develop an S^1-action on fibres. This twisted S^1-action will allow us to develop a Delzant-type correspondence, where when we compare two manifolds, if the S^1-spaces arising via bundle fibres are symplectomorphic via the twisted S^1-action, then the corresponding Hamiltonian SO(3)-spaces are equivariant symplectomorphic. In this section, we will:

1. Develop a framework for the twisted S^1-action.
2. Prove a version of Delzant's Theorem.

6.1.1 Construction of the Twisted S^1-Action

Recall how the transition functions arising from the π bundle acted. On each fibre $\pi^{-1}(\mu)$ there is the group action G_μ. Now, we consider the group isomorphism:

$$k_\mu : S^1 \to G_\mu \text{ where } \mu = g \cdot \mu_0 \qquad \text{given by } k_\mu(t) \mapsto gtg^{-1} \tag{6.1}$$

This allows us to define an T-action $\bullet$ on P given by:

Definition 6.1.1 (T-action on P). *Let $(P, \omega, \mathrm{SO}(3))$ be a symplectic manifold with Hamiltonian Lie action. Then, a twisted T action $T \circlearrowleft P$ is given by:*

$$t \bullet x = k_{\pi(x)}(t) \cdot x \tag{6.2}$$

Where $k_\mu(t) \mapsto gtg^{-1}$, and T is the maximum tori of SO(3).

Remark 6.1.1. *We say that T acts on x via a twisted action $k_{\pi(x)}$, which is manifestly abelian. By definition, the generators of this action are:*

Definition 6.1.2 (Infinitesimal generators of T action)**.** *The infinitesimal generators are:*

$$\xi_P = \lim_{\tau \to 0} \frac{d}{d\tau} \exp(\tau\xi) \bullet x \tag{6.3}$$

Lemma 6.1.1 (Generators of T-action)**.** *The infinitesimal generators of $T = S^1$ and $G = \mathrm{SO}(3)$ are related for any $\xi \in \mathfrak{t} \subset \mathfrak{g}$ by:*

$$\xi_P^T(x) = (\mathrm{Ad}_g \xi)_P(x) \tag{6.4}$$

Proof. This follows by derivation of the above $\bullet$ definition. Take the derivative of $\exp(t\xi) \cdot x$:

$$\xi_P^T = \lim_{\tau \to 0} \frac{d}{d\tau} g \exp(\tau\xi) g^{-1} \tag{6.5}$$

$$\xi_P^T = \mathrm{Ad}_g \xi \tag{6.6}$$

Where $\mu = g \cdot \mu_0$, or $\pi(x) = g \cdot \mu_0 \iff g^{-1} \cdot x \in \pi^{-1}(\mu_0)$. ☐

Theorem 6.1.2 (Blaom (1996))**.** *The T action defined above is Hamiltonian. Furthermore, a T-equivariant moment map for this action is $j = i \circ \pi_{\mathcal{W}} \circ J$*

Proof. We want to show:

$$\langle dj_{x'} v, \xi \rangle = \langle dJ_{x'} v, \mathrm{Ad}_g \xi \rangle \tag{6.7}$$

when $v \in \mathrm{Hor}_x$ and when $v \in T_x \pi^{-1}(\mu)$. As these are ω-orthogonal, that will imply that:

$$\langle dj_{x'} v, \xi \rangle = \langle dJ_{x'} v, \mathrm{Ad}_g \xi \rangle \tag{6.8}$$

everywhere. Moreover, this will prove that T is Hamiltonian with the infinitesimal generators:

$$X_{j_\xi} = \xi_P^T(x) \tag{6.9}$$

Let's prove the first part - note that j is G-invariant, since orbits factor out under $\pi_{\mathcal{W}}$. Since the $T \trianglelefteq G$ action is trivial, we also have j is T-equivariant. For fixed μ_0, choose $\mu = \pi(x) = g \cdot \mu_0$, and note that for the Weyl chamber $\mathcal{W}$, $g^{-1} J(x') \in \mathcal{W}$. Now, for any $\xi \in \mathfrak{t}$, we note that $g^{-1} J(x')$ survives the projection $\pi_{\mathcal{W}}$.

$$\langle j(x'), \xi \rangle = \langle \pi_{\mathcal{W}} J(x'), \xi \rangle = \langle g^{-1} \cdot J(x'), \xi \rangle \tag{6.10}$$

Taking derivatives of both sides, we have:

$$\langle dj_{x'} v, \xi \rangle = \langle dJ_{x'} v, \mathrm{Ad}_g \xi \rangle \tag{6.11}$$

In particular, this occurs when $x' \in T_x \pi^{-1}(\mu)$. ☐

Lemma 6.1.3. *Both the derivative of j and the derivative of J vanish on horizontal fibres:*

$$\{\langle dJ_{x'}v, \mathrm{Ad}_g\xi\rangle : v \in \mathrm{Hor}_x\} = 0 \tag{6.12}$$

Proof. Firstly, since j is G-invariant, yet elements of Hor_x live in the domain $T_x(G \cdot x)$, we conclude that $\{\langle dj_{x'}v, \xi\rangle : v \in \mathrm{Hor}_x\} = 0$. Now, to show that the derivative of J vanishes, note that $dJ_{x'}v$ must be tangent to the coadjoint orbit through x, so that: $dJ_{x'}v = \mathrm{ad}_\nu J(x')$. Then:

$$\begin{aligned}\langle dJ_{x'}v, \mathrm{Ad}_g\xi\rangle &= \langle \mathrm{ad}_\nu J(x'), \mathrm{Ad}_g\xi\rangle && (6.13)\\ &= \langle J(x'), [\nu, \mathrm{Ad}_g\xi]\rangle && (6.14)\\ &= \langle \mathrm{Ad}^*_g J(x'), [\mathrm{Ad}_{g^{-1}}\nu, \xi]\rangle && (6.15)\\ &= -\langle \mathrm{ad}^*_\xi \left[g^{-1} \cdot J(x')\right], \mathrm{Ad}_{g^{-1}}\nu\rangle && (6.16)\end{aligned}$$

Since $g^{-1}J(x') \in \mathcal{W}$, $\mathrm{ad}^*_\xi\left[g^{-1} \cdot J(x')\right] = 0$, the conclusion follows. □

Lemma 6.1.4 (Classical)**.** *If G is a Lie group and H a closed subgroup of G, then $G \to G/H$ is a principal G bundle.*

We will use this lemma to build charts for a symplectomorphism on the $G = \mathrm{SO}(3)$ level between manifolds.
References: Blaom (1996)

6.1.2 Equivalence of Hamiltonian G-Spaces

Theorem 6.1.5. *Let G be a compact, connected Lie group. Then two Hamiltonian $G =$ SO(3)-spaces with regular momenta are equivalent if and only if their associated T-spaces are equivalent.*

Proof. The forward direction is clear. Let (P_1, ω_1, G, J^1) and (P_2, ω_2, G, J^2) be Hamiltonian G-spaces. We have symplectic fibrations $\pi_1 : P_1 \to \mathcal{O}$ and $\pi_2 : P_2 \to \mathcal{O}$. Let F_1 and F_2 be fibres of π_1 and π_2 over μ_0 respectively, where μ_0 is just a choice $\mu_0 \in \mathfrak{g}^*_{reg}$. Let's assume we have a T-equivariant symplectomorphism:

$$\begin{aligned}\varphi : F_1 \to F_2 && (6.17)\\ J^{F_2} \circ \varphi = J^{F_1} && (6.18)\end{aligned}$$

Let $x \in P_1$ such that $\pi_1(x) = g \cdot \mu_0$. Define:

$$\phi(x) = g \cdot \varphi(g^{-1} \cdot x) \tag{6.19}$$

It suffices to show that ϕ is well-defined, ϕ is G-equivariant and $J^2 \circ \phi = J^1$, and there exists an inverse for ϕ. □

Lemma 6.1.6. *$\phi(x)$ is well defined, for a choice of $g \in G$.*

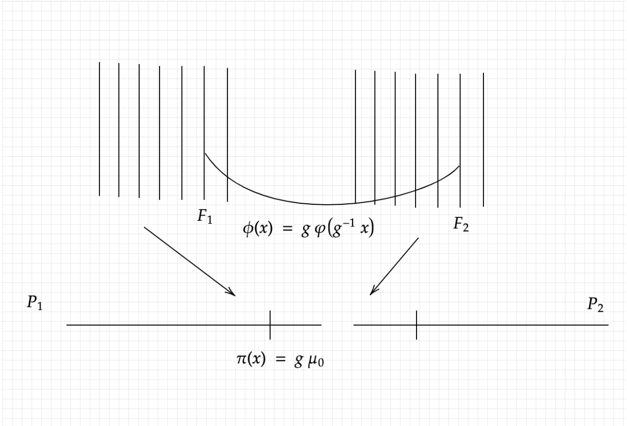


Figure 6.1: To move between fibres, we define $\phi(x) = g \cdot \varphi(g^{-1} \cdot x)$

Proof. We consider g such that $g \cdot \mu_0 = \mu_0$. Note that $g \cdot \mu_0$ is unique up to a factor of $g \cdot t$, where $t \in G_{\mu_0}$. Now we note that $\phi(x)$ does not depend on g:

$$\phi(x) = gt \cdot \varphi(t^{-1} \circ g^{-1} \cdot x) \tag{6.20}$$

$$\phi(x) = g \cdot \varphi(g^{-1} \cdot x) \tag{6.21}$$

□

Lemma 6.1.7. *ϕ is G-equivariant.*

Proof. Since $\pi_1(x) = g \cdot \mu_0$, $\pi_1(h \cdot x) = hg \cdot \mu_0$ by equivariance. Now:

$$\phi(h \cdot x) = hg \cdot \varphi(g^{-1}h^{-1} \cdot hx) = h \cdot \phi(x) \tag{6.22}$$

□

Lemma 6.1.8. *$J^{F_2} \circ \varphi = J^{F_1}$ implies that $J^2 \circ \phi = J^1$.*

Proof. We do the following computation:

$$\begin{aligned}
(J^2 \circ \phi)(x) &= J^2(g \cdot \varphi(g^{-1} \cdot x)) && (6.23)\\
&= g \cdot J^{F_2}(\varphi(g^{-1} \cdot x)) && (6.24)\\
&= g \cdot J^{F_1}(g^{-1} \cdot x) && (6.25)\\
&= J^1(x) && (6.26)
\end{aligned}$$

□

Lemma 6.1.9. *There exists an inverse, ϕ^{-1} for ϕ.*

Proof. This follows by symmetry of definitions:

$$\phi^{-1}(y) = g \cdot \varphi^{-1}(g^{-1} \cdot y) \tag{6.27}$$

Note that:

$$\phi^{-1}(\phi(x)) = g \cdot \varphi^{-1}(g^{-1} \cdot g \cdot \varphi(g^{-1} \cdot x)) = x \tag{6.28}$$

□

Now, by hypothesis:

$$\varphi : F_1 \to F_2 \tag{6.29}$$

is a symplectic diffeomorphism of two fibres over μ_0. Equivalently, this is a diffeomorphism of the kernel of $d\pi_1$ and $d\pi_2$ around points $x \in P_1,\ \varphi(x) \in P_2$:

$$\varphi : \ker(d\pi_1)_x \to \ker(d\pi_2)_{\varphi(x)} \tag{6.30}$$

Since $\mathrm{Hor}^2_{\phi(x)} = (d\pi_2)^{\omega}_{\varphi(x)}$, and $\mathrm{Hor}^1_x = (d\pi_1)^{\omega}_x$, it follows $d\phi$ maps Hor^1_x to $\mathrm{Hor}^2_{\phi(x)}$. Furthermore, since each is the ω-complement, the isomorphism is symplectic. So, ϕ is a symplectic diffeomorphism.
References: Blaom (1996)

6.1.3 Delzant Correspondence for SO(3)

Theorem 6.1.10 (Delzant (1988) and Blaom (1996))**.** *Consider two* 4*-dimensional, compact Hamilitonian* SO(3) *spaces* $(P_1, \omega_1, \mathrm{SO}(3), J^1)$ *and* $(P_2, \omega_2, \mathrm{SO}(3), J^2)$ *whose moment images do not contain* 0*. Denote* $S = S^1_{xy}$ *the circle in the xy-plane. If rotation actions in* S *are faithful on* $(J^i)^{-1}(\hat{z})$ *then the spaces are equivalent up to equivariant symplectomorphism.*

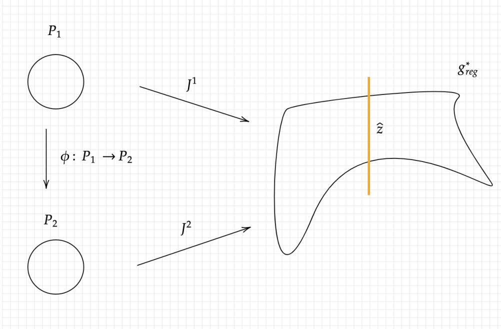


Figure 6.2: Existence of a SO(3)-equivariant symplectomorphism ϕ comes from agreement of two moment maps, J^2 and J^1, such that the inverse images of $\hat{z}$ are acted on faithfully.

Proof. Let $(P_1, \omega_1, \mathrm{SO}(3), J^1)$ and $(P_2, \omega_2, \mathrm{SO}(3), J^2)$ be two 4-dimensional, compact Hamiltonian SO(3) spaces whose moment-images do not contain 0. Let F_1 be a fibre of π_1. Then $\dim F_1 + \dim \mathcal{O} = \dim P_1$ and $\dim F_2 + \dim \mathcal{O} = \dim P_2$. By Delzant's Theorem, we have that that $\dim F_1 = 2 \dim S^1 = 2$. Furthermore, $\dim \mathcal{O} = 2$. Therefore, we require that $\dim P_1 = \dim P_2 = 4$. By *Delzant's Theorem*, because both S^1 rotation actions are faithful with respect to the twisted action, the Delzant spaces $S^1 \circlearrowright P_1$ and $S^1 \circlearrowright P_2$ are symplectomorphic. By Theorem 6.1.5, the SO(3) spaces are equivalent up to equivariant symplectomorphism by an element of SO(3). $\square$

References: Delzant (1988) and Blaom (1996)

Chapter 7

Necessity of Regular Momenta

7.1 Necessity of the Regular Condition

Let's study the theorem in the restriction when the image of J is no longer regular. Using the Fubini-Study metric, we are able to find the image of the moment map J, arising from the Lie group action of SO(3) on $(\mathbb{C}P^2, \omega_{FS})$, and study the unique properties of this construction.

7.1.1 Fubini-Study Metric and Symplectic Structure

To develop this construction, let V be a complex vector space, with dimension $\dim_{\mathbb{C}} V = n+1$ with a Hermitian inner product $\langle \cdot, \cdot \rangle$ and $\pi : V \setminus \{0\} \to \mathbb{P}V$.

Lemma 7.1.1 (Symplectic form on V)**.** *V is a vector space with symplectic form $\Omega(v, w) = \operatorname{Im}\langle v, w \rangle$*

Proof. We verify that Ω is skew-symmetric:

$$\Omega(w, v) = \operatorname{Im}\langle w, v \rangle = \operatorname{Im} \overline{\langle v, w \rangle} = -\operatorname{Im}\langle v, w \rangle = -\Omega(v, w) \tag{7.1}$$

Where we use conjugate symmetry to get that $\langle v, w \rangle = \overline{\langle w, v \rangle}$. Replacing w with iw, this form is also non-degenerate since if $\Omega(v, \cdot) = 0$, then $\operatorname{Im}\langle v, w \rangle = 0$ and for choices of iw, $\operatorname{Re}\langle v, w \rangle = 0$ as well, showing that $v = 0$ using that $\langle \cdot, \cdot \rangle$ is non-degenerate. ☐

Remark 7.1.1. *Now, we want to find a symplectic form ω_{FS} for $(\mathbb{P}V, \omega_{FS})$*

Note that in the construction $\pi : V \setminus \{0\} \to \mathbb{P}V$, orthogonal complements survive in the projectivization: $d\pi_v$ maps $(\mathbb{C}v)^\perp$ isomorphically onto $T_{\pi(v)}\mathbb{P}V$. Therefore, we work backwards, and consider a symplectic built from the orthogonal complements of ξ_1, ξ_2 with respect to v on V. These are $\xi_i - \frac{\langle \xi_i, v \rangle}{\langle v, v \rangle} v \in (\mathbb{C}v)^\perp$ for $i \in \{1, 2\}$.

Remark 7.1.2. *Therefore, we suspect that*

$$\omega_v(\zeta_1, \zeta_2) = \operatorname{Im} \left\langle \xi_1 - \frac{\langle \xi_1, v \rangle}{\langle v, v \rangle} v, \xi_2 - \frac{\langle \xi_2, v \rangle}{\langle v, v \rangle} v \right\rangle$$

would be a symplectic form for $\mathbb{P}V$*. However, this form doesn't yet survive in the quotient. Consider the diffeomorphism given by* $m_\lambda : V \setminus \{0\} \to V \setminus \{0\}$ *given by* $m_\lambda(v) = \lambda v$*. We note that we need* $m_\lambda^* \omega = \omega$*. Yet, observe that:* $m_\lambda^* \omega_v = |\lambda|^2 \omega_v(\zeta_1, \zeta_2)$*. It suffices to take the Fubini-Study metric to be a normalized* $\frac{1}{\langle v,v\rangle}$ *expression of the proposed two-form. The reason for this normalization is that* $\langle \lambda v, \lambda v\rangle$ *cancels out the* $|\lambda|^2$ *term.*

Lemma 7.1.2 (Symplectic form on $\mathbb{P}V$)**.** $(\mathbb{P}V, \omega_{FS})$ *is a symplectic manifold with:*

$$(\omega_{FS})_{[v]}(\zeta_1, \zeta_2) = \frac{1}{\langle v, v\rangle} \operatorname{Im} \left\langle \xi_1 - \frac{\langle \xi_1, v\rangle}{\langle v, v\rangle} v, \xi_2 - \frac{\langle \xi_2, v\rangle}{\langle v, v\rangle} v \right\rangle \tag{7.2}$$

where $\xi_j \in V$ *are such that* $d\pi_v \xi_j = \zeta_j$

Now note that ω_{FS} is well defined and, mimicking the above argument, it is non-degenerate, while closedness follows from $PU(V, \langle \cdot, \cdot \rangle)$ invariance according to Arnold (2010).

References: Arnold (2010)

7.1.2 Moment Map for Unitary Representations

We are now able to apply Proposition 1.1 in Wildberger (1992).

Lemma 7.1.3 (Wildberger (1992))**.** *If* $G \to U(V, \langle \cdot, \cdot \rangle)$ *is a unitary representation of a Lie group* G*, the induced action of* G *on* $\mathbb{P}V$ *is Hamiltonian, with moment map:*

$$J^X([v]) = \frac{1}{2i} \frac{\langle X \cdot v, v\rangle}{\langle v, v\rangle} \tag{7.3}$$

for $[v] \in \mathbb{P}V$ *and* $X \in \mathfrak{g}$*.*

References: Wildberger (1992)

7.1.3 Moment Map for SO(3) Action on $\mathbb{C}P^2$

Now, consider the action of $G = \mathrm{SO}(3)$ on $V = \mathbb{C}^3$ given by evaluations, that is: $A \cdot z = Az$, where A is the $\mathbb{C}$-linear extension of $A : \mathbb{R}^3 \to \mathbb{R}^3$. In other words, we compose with the inclusion SO(3) into SU(3). The Hermitian inner product in $\mathbb{C}^3$ is:

$$\langle (z_1, z_2, z_3), (w_1, w_2, w_3)\rangle = z_1\overline{w_1} + z_2\overline{w_2} + z_3\overline{w_3} \tag{7.4}$$

Now, we compute the moment image of the Hamiltonian action of SO(3) on $(\mathbb{C}P^2, \omega_{FS})$. With the usual identification of $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ we note that $J : \mathbb{C}P^2 \to \mathbb{R}^3$. We want to use the moment map formula to compute the moment-image of the resulting Hamiltonian action of SO(3) on $(\mathbb{C}P^2, \omega_{\mathrm{FS}})$ Wildberger (1992). With the usual identification of $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$ in place.

Lemma 7.1.4 (Moment Map μ for $\mathbb{C}P^2$)**.** *We claim the moment map μ is:*

$$\mu\big([z_1 : z_2 : z_3]\big) = \left(\frac{\mathrm{Im}(z_2\overline{z_3})}{|z_1|^2+|z_2|^2+|z_3|^2}, \frac{\mathrm{Im}(z_3\overline{z_1})}{|z_1|^2+|z_2|^2+|z_3|^2}, \frac{\mathrm{Im}(z_1\overline{z_2})}{|z_1|^2+|z_2|^2+|z_3|^2}\right). \tag{7.5}$$

Proof. The following are bases on $\mathfrak{so}(3)^*$ with the usual identification:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.6}$$

To show the formula for the moment map, we apply Wildberger's formula to each of the basis elements and use linearity:

$$E_1 z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -z_3 \\ z_2 \end{pmatrix}, \tag{7.7}$$

so that

$$\begin{aligned} \frac{1}{2i}\,\frac{\langle E_1 z,\, z\rangle}{\langle z,\, z\rangle} &= \frac{1}{2i}\,\frac{\big(-z_3\,\overline{z_2} + z_2\,\overline{z_3}\big)}{|z_1|^2+|z_2|^2+|z_3|^2} && (7.8)\\ &= \frac{1}{2i}\,\frac{2\,i\,\mathrm{Im}(z_2\overline{z_3})}{|z_1|^2+|z_2|^2+|z_3|^2} = \frac{\mathrm{Im}(z_2\overline{z_3})}{|z_1|^2+|z_2|^2+|z_3|^2}. && (7.9) \end{aligned}$$

Similarly, we have that

$$\frac{1}{2i}\,\frac{\langle E_2 z,\, z\rangle}{\langle z,\, z\rangle} = \frac{\mathrm{Im}(z_3\overline{z_1})}{|z_1|^2+|z_2|^2+|z_3|^2} \quad \text{and} \quad \frac{1}{2i}\,\frac{\langle E_3 z,\, z\rangle}{\langle z,\, z\rangle} = \frac{\mathrm{Im}(z_1\overline{z_2})}{|z_1|^2+|z_2|^2+|z_3|^2}. \tag{7.10}$$

□

7.1.4 Significance for Delzant Correspondence

Remark 7.1.3. *Therefore, our formula for the moment map holds. Yet, the action does not have regular momenta, since $\mu([0 : 0 : 1]) = (0, 0, 0)$ is the origin. As we have proven, this means that the image of J is a closed ball. Furthermore, as we will see below, another space which has the closed ball as its moment map image is $S^2(r) \times S^2(r)$. Therefore, we see that regular momenta is a necessary assumption in the statement of our theorem. Without it, we would have concluded that $S^2(r) \times S^2(r) \cong \mathbb{C}P^2$. But this is impossible, as the cohomology is wrong: $H^2_{dR}(\mathbb{C}P^2) = \mathbb{R}$ while $H^2_{dR}(S^2 \times S^2) = \mathbb{R}^2$.*

Chapter 8

Classification of Hamilitonian SO(3)-actions

8.1 Action Classification and Realization Theorems

In general, we can realize each of the four prototypes, and therefore, we have a classification theorem for spaces admitting effective, Hamiltonian SO(3) actions, with regular momenta. To do so, we use Lemma 8.1.1. In this section we

1. Prove 8.1.1
2. Give the classification theorem

8.1.1 Classification of Moment Images

Lemma 8.1.1. *Whenever* $r > 0$*, the action of* SO(3) *on* $S^2(r)$ *given by evaluations is Hamiltonian, with moment map given by the inclusion* $S^2(r) \hookrightarrow \mathbb{R}^3$ *Marsden and Weinstein (1974).*

Proof. Since the coadjoint action of SO(3) on $\mathbb{R}^3$ is evaluation, the infinitesimal vector fields are $X_a^\# = \phi^{-1}(X) \times a$ where $\phi\colon \mathbb{R}^3 \to \mathfrak{so}(3)$ and $X \in \mathfrak{so}(3)$, and $a \in \mathbb{R}^3$. Recall the triple-product identity when $x \perp v$: $(a \times x) \times v = -x(a \cdot v)$. Then we note that $x \cdot ((a \times x) \times v) = -x \cdot (x(a \cdot v)) = -a \cdot v$. As we made the canonical isomorphism, we are in $\mathbb{R}^3$, where ω_x is the area form: $\omega_x(v, w) = x \cdot (v \times w)$. Therefore: $\omega_x(X_a, v) = a \cdot v$. Let $\mu(x) = x$. Then:

$$d\langle \mu(x), a\rangle = a \cdot v \qquad\qquad d\mu_x^a(v) = \omega_x(X_a, v) \tag{8.1}$$

This verifies that μ is indeed a moment map. $\square$

Now, according to Lemma 8.1.1, and using Lemma 4.1.1, we have the following prototypes of our classification theorem, therefore, we declare using these two lemmas:

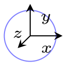
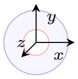
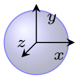
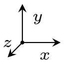
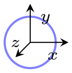
dim	**F?**	M	**Action**	$\mu(M)$
2	Y	$S^2(r)$	$A \cdot p = Ap$	 **Sphere** $S^2(r)$ $\{x \in \mathbb{R}^3 : \|x\| = r\}$
4	Y	$S^2(r_1) \times S^2(r_2)$	$A \cdot (p, q) = (Ap, Aq)$	 **Hollow ball** $\{x \in \mathbb{R}^3 : r_2 - r_1 \leq \|x\| \leq r_1 + r_2\}$
4	Y	$S^2(r) \times S^2(r)$	$A \cdot (p, q) = (Ap, Aq)$	 **Closed ball** $\{x \in \mathbb{R}^3 : \|x\| \leq 2r\}$
2	N	S^2	$Ap = p$	 **Origin point** $\{0\}$
4	Y	$S^2(r) \times S^2$	$A \cdot (p, q) = (Ap, q)$	 **Sphere** $S^2(r)$

Table 8.1: Moment map images for different manifolds and group actions

Remark 8.1.1. *Note that there are several faithful cases.*

Theorem 8.1.2. *Any compact, connected Hamiltonian* SO(3)*-space with regular momenta has a moment map whose image is one of the following* SO(3)*-invariant subsets of* $\mathbb{R}^3$*:*

1. *A sphere* $S^2(r)$ *of radius* $r > 0$*,*
2. *A closed hollow ball* $\{x \in \mathbb{R}^3 : r_1 \leq \|x\| \leq r_2\}$ *where* $0 < r_1 < r_2$*,*
3. *A closed ball* $\{x \in \mathbb{R}^3 : \|x\| \leq r\}$ *where* $r > 0$*, or*
4. *The origin* $\{0\}$*.*

Moreover, for each of these possibilities, there exists a Hamiltonian SO(3)*-space whose moment map has precisely this image.*

This classification theorem, combined with our generalized Delzant correspondence (Theorem 6.1.10), provides a complete characterization of Hamiltonian SO(3)-spaces with regular momenta, up to equivariant symplectomorphism. As shown in the previous section, the regularity condition is essential, as without it, non-equivalent spaces such as $\mathbb{CP}^2$ and $S^2(r) \times S^2(r)$ can have identical moment map images.

References: Delzant (1988) and Melvin and Parker (1986).

8.1.2 Areas of Future Inquiry

We have that 4-manifolds admitting effective SO(3)-actions, but it's not easy to figure out which of them (beyond the ones we have previously mentioned) may admit symplectic forms. Melvin and Parker (1986) described a list of manifolds supporting Hamiltonian, effective actions:

1. S^4 or $\pm\mathbb{C}P^2$
2. Connected sums of copies of $S^1 \times S^3$ and $S^1 \times P^3$
3. $(SU(2)/H)$-bundles over S^1 $\quad$ (H a finite subgroup of $SU(2)$)
4. S^2-bundles over surfaces
5. Certain quotients of S^2-bundles over surfaces by involutions.

Yet due to cohomological restrictions, many of these do not support a symplectic form.

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Appendix A

Elements of Differential Topology

:

A.1 Review of Differential Topology

In this appendix, we try to provide the reader a short introduction to differential topology, sufficient to read the thesis.

A.1.1 Vectors, Covectors and Vector Bundles

In order to define the notion of tangent spaces, and cotangent spaces, we first define the notion of function germs. This definition captures the behavior of a function f around a point p. The following presentation is from Dundas (2018)

Definition A.1.1 (A Function Germ)**.** *Let M, N be smooth manifolds, and consider $p \in M$. We say that functions $f : M \to N$ and $g : M \to N$ are equivalent $f \sim g$, if there exists an open neighborhood of p, U_{fg} such that $f = g$ on U_{fg}. In that case, we denote the equivalence class of f, $[f] = \overline{f}$, and the function germ $\overline{f}$ is:*

$$\overline{f} : (M, p) \to (N, f(p))$$

Now, the tangent space to M at p is the equivalence class of function germs passing through p with a given velocity:

Definition A.1.2 (Tangent space to M at p, T_pM)**.** *Let M be a smooth n-dimensional manifold. Let $p \in M$.*

$$W_p = \{\text{germs } \overline{\gamma} : (\mathbb{R}, 0) \to (M, p)\}$$

Two germs $\overline{\gamma}, \overline{\gamma}_1 \in W_p$ are equivalent if two function germs $\overline{\phi} : (M, p) \to (\mathbb{R}, \phi(p))$ we have that $(\phi\gamma)'(0) = (\phi\gamma_1)'(0)$. We define the tangent space to M at p to be the set of equivalence classes

$$T_pM = W_p/\sim$$

Remark A.1.1. *Therefore, the tangent space is best thought of as an equivalence class of curves*

As above, let M be a smooth manifold, and consider $p \in M$. We can define the cotangent space T_p^*M as the vector space dual to T_pM.

Definition A.1.3 (Dual Vector Space)**.** *Given a real vector space V, we define V^* to be the vector space* $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$*, the set of* $\mathbb{R}$*-linear maps from* $V \to \mathbb{R}$

Definition A.1.4 (Cotangent Space)**.** *As above, let M be a smooth manifold, and consider $p \in M$. Given the tangent vector space T_pM, the cotangent space, T_p^*M contains all linear maps p of the form:*

$$T_p^*M = \{p : V \to \mathbb{R}, p \text{ linear } \}$$

Now, given the tangent spaces and cotangent spaces, there are two set-theoretic constructions we often make: $T^*M = \bigsqcup_{m\in M} T_m^*M$ and $TM = \bigsqcup_{m\in M} T_mM$. These are sets which are disjoint unions of all tangent planes as we index $m \in M$. These sets are known as the cotangent and tangent bundles, respectively.

Definition A.1.5 (A Real Vector Bundle)**.** *A real vector bundle of rank n is a pair (E, X), where E is a topological space space called the total space, and X is a topological space called the base space, together with a continuous map $\pi : E \to X$ such that:*

1. *For each $p \in X$, the fibre $\pi^{-1}(p)$ is a n-dimensional real vector space.*
2. *For every point $p \in X$, there exists an open neighborhood $U \subset X$ containing p, and a homeomorphism*

   $$h : \pi^{-1}(U) \to U \times \mathbb{R}^n$$

   such that the following diagram commutes:

   $$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "h"] \arrow[dr, "\pi|_{\pi^{-1}(U)}"'] & & U \times \mathbb{R}^n \arrow[dl, "proj_1"] \\ & U & \end{tikzcd}$$

   In other words, $\pi = proj_1 \circ h$.
3. *For each $q \in U$, the map h is to a vector space isomorphism on fibres:*

   $$h_q : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n,$$

Remark A.1.2. *A real vector bundle is a fibreing over the base space by vector spaces, such that we can move through the fibreing continuously via h.*

Remark A.1.3. *T^*M and TM are both vector bundles*

References: Terek (2020), Dundas (2018)

A.1.2 Tensor Product

Given two vector real spaces V and W, the tensor product of V and W is:

$$V \otimes W = (V \times W)/S$$

where S is the linear subspace generated by linear combinations of

$$e_1 = (v_1 + v_2, w) - (v_1, w) - (v_2, w) \tag{A.1}$$
$$e_2 = (v, w_1 + w_2) - (v, w_1) - (v, w_2) \tag{A.2}$$
$$e_3 = (sv, w) - s(v, w) \tag{A.3}$$
$$e_4 = (v, sw) - s(v, w) \tag{A.4}$$

Definition A.1.6 (Tensor Product of Linear Maps). *The tensor product of two linear maps $A : V \to V$ and $B : W \to W$ is the unique linear map such that:*

$$(A \otimes B) : (V \otimes W) \to (V \otimes W)$$

where $s \in \mathbb{R}$, $(v, w) \in V \times W$.

References: Isham (1999)

A.1.3 Exterior Derivative, Closed and Exact Forms

Definition A.1.7. *An n-form is a tensor field ω of type $(0, n)$ that is totally skew-symmetric, in the sense that, for any permutation P of the indices $1, 2, \dots, n$:*

$$\omega(X_1, \dots, X_n) = (-1)^{\deg P} \omega(X_{P(1)}, \dots, X_{P(n)})$$

In particular, the $\wedge$ product is the tool which defines multiplication of n_1-forms by n_2-forms and outputs $(n_1 + n_2)$-forms. It is the antisymmetric part of the tensor product of both forms:

$$\omega_1 \wedge \omega_2 = \frac{1}{n_1! n_2!} \sum_{\sigma \in P} (-1)^{\deg \sigma} (\omega_1 \otimes \omega_2)^\sigma$$

Definition A.1.8 (Exterior Derivative d). *The exterior derivative of ω is the $(n+1)$-form $d\omega$ defined by:*

$$d\omega(X_1, \dots, X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} X_i(\omega(X_1, \dots, \hat{X}_i, \dots, X_{n+1})) \tag{A.5}$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{n+1}) \tag{A.6}$$

Remark A.1.4. *In the case of* 2*-forms, we have that:*

$$d\omega(X,Y) = X(\langle \omega, Y\rangle) - Y(\langle \omega, X\rangle) - \langle \omega, [X,Y]\rangle$$

In the DeRham Theory, we consider the DeRham complex, a sequence of vector spaces:

$$0 \xrightarrow{d} C^{\infty}(\mathcal{M}) \xrightarrow{d} A^1(\mathcal{M}) \xrightarrow{d} A^2(\mathcal{M}) \xrightarrow{d} \ldots \xrightarrow{d} A^m(\mathcal{M}) \xrightarrow{d} 0$$

Notably, since $d^2 = 0$, one has:

$$\mathrm{Im}(d : A^{n-1} \to A^n) \subset \mathrm{Ker}(d : A^n \to A^{n+1})$$

An n-form ω is closed if $d\omega = 0$. We define the set of all closed n-forms as:

$$Z^n(\mathcal{M}) = \mathrm{Ker}(d : A^n \to A^{n+1}) = \{\omega \in A^n(\mathcal{M}) \mid d\omega = 0\}$$

Similarly, the set of all exact forms is:

$$B^n = \mathrm{Im}(d : A^{n-1} \to A^n) = \{\omega \in A^n(\mathcal{M}) \mid \omega = d\beta \text{ for some } \beta \in A^{n-1}(\mathcal{M})\}$$

$$H^n_{DR}(\mathcal{M}) = Z^n(\mathcal{M})/B^n(\mathcal{M})$$

References: Isham (1999)

A.2 Additional Topics

A.2.1 Derivation of A Lie Bracket Identity

Definition A.2.1 (Lie Derivative of a Vector Field)**.** *For a smooth manifold M, consider $X, Y \in \mathfrak{X}(M)$. Then, the Lie derivative of X, in the Y direction is:*

$$\mathcal{L}_Y(X) = [Y, X]$$

Where $[X, Y]$ denotes the vector field commutator of X and Y.

For the following results, we apply Cartan's Magic Formula:

Theorem A.2.1 (Cartan's Magic Formula)**.**

$$\mathcal{L}_X = d(i_X) + i_X d$$

Lemma A.2.2 (Fundamental Contraction Identity)**.** *Let $\mathcal{L}$ be the Lie-derivative, i denote contraction, and d denote the exterior derivative. Then:*

$$[\mathcal{L}_X, i_Y] = i_{[X,Y]}$$

Proof. Expand $\mathcal{L}_X$ according to the Lie-bracket:

$$[\mathcal{L}_X, i_Y] = [d \circ i_X + i_X \circ d, i_Y] = [i_X \circ d, i_Y] = i_X \mathcal{L}$$

As $[i_X, i_Y] = 0$. Then we use the fact that:

$$i_x \circ di_Y - i_Y \circ i_X \circ d = i_x \circ di_Y - i_Y \circ i_X \circ d = i_X(di_Y + i_Y d) = i_X \mathcal{L}_Y$$

□

Now using this lemma, we prove the following:

Lemma A.2.3.

$$i_{[X_f, X_g]}\omega = \mathcal{L}_{X_f} i_{X_g}\omega$$

Proof. Using A.2.2 we have that:

$$i_{[X_f, X_g]}\omega = [\mathcal{L}_{X_f}, i_{X_g}]$$

Then:

$$i_{[X_f, X_g]}\omega = \mathcal{L}_{X_f}(dg) = d(X_f(g)) = -\omega(X_f, X_g)$$

Therefore, we conclude. □

References: Cannas da Silva (2001), Isham (1999)

A.2.2 Example with Fubini Study Metric on $\mathbb{C}P^1$

Example A.2.1 (Fubini-Study Metric on $\mathbb{C}P^1$)**.** *The Fubini-Study metric is the area form on a sphere S^2 when $V = \mathbb{C}^2$, and hence $\mathbb{P}V = \mathbb{C}P^1$. For $\mathbb{C}P^1$, we use homogeneous coordinates $[z_0 : z_1]$. In the chart where $z_0 \neq 0$, we can use the coordinate $w = z_1/z_0$. A point in $\mathbb{C}P^1$ is represented as $v = (1, w)$ in this chart, with inner product: $\langle v, v\rangle = 1 + |w|^2$. Let's denote our two tangent vectors as: $\zeta_1 = (0, \xi_1)$, $\zeta_2 = (0, \xi_2)$. Now, $\langle \zeta_1, \zeta_2\rangle = \overline{\xi_1} \cdot \xi_2 = \overline{\xi_1}\xi_2$ and $\langle \zeta_1, v\rangle = \overline{\xi_1} \cdot w = \overline{\xi_1}w$ and $\langle v, \zeta_2\rangle = \overline{w} \cdot \xi_2 = \overline{w}\xi_2$. Then, the form is:*

$$\omega_{[v]}(\tilde{\zeta_1}, \tilde{\zeta_2}) = Im\left(\frac{(\overline{\xi_1}\xi_2)(1 + |w|^2) - (\overline{\xi_1}w)(\overline{w}\xi_2)}{(1 + |w|^2)^2}\right) \tag{A.7}$$

Note that the following hold:

$$\begin{aligned}
(\overline{\xi_1}\xi_2)(1 + |w|^2) - (\overline{\xi_1}w)(\overline{w}\xi_2) &= \overline{\xi_1}\xi_2 + \overline{\xi_1}\xi_2|w|^2 - \overline{\xi_1}w\overline{w}\xi_2 && \text{(A.8)}\\
&= \overline{\xi_1}\xi_2 + \overline{\xi_1}\xi_2|w|^2 - \overline{\xi_1}\xi_2|w|^2 && \text{(A.9)}\\
&= \overline{\xi_1}\xi_2 && \text{(A.10)}
\end{aligned}$$

Note that $w\overline{w} = |w|^2$, *which is why* $\overline{\xi_1} w\overline{w}\xi_2 = \overline{\xi_1}\xi_2|w|^2$.

$$\omega_{[v]}(\tilde{\zeta_1}, \tilde{\zeta_2}) = Im\left(\frac{\overline{\xi_1}\xi_2}{(1+|w|^2)^2}\right) \tag{A.11}$$

$$Im(\overline{\xi_1}\xi_2) = \frac{1}{2i}(\overline{\xi_1}\xi_2 - \xi_1\overline{\xi_2}) \tag{A.12}$$

Note that the expression for vectors $\overline{\xi_1}\xi_2 - \xi_1\overline{\xi_2}$ *implies that the symplectic form must be antisymmetric and contain a factor* $-dw \wedge d\overline{w}$. *This can be written in terms of differential forms as:*

$$\omega_{[v]} = \frac{i/2}{(1+|w|^2)^2} dw \wedge d\overline{w} \tag{A.13}$$

Writing $w = x+iy$, *we have that:* $\omega_{FS} = \frac{1}{(1+x^2+y^2)^2} dx \wedge dy$. *Compare this to the expression for the area form of* S^2 *in* $\mathbb{R}^2$ *using stereographic projection:* $\omega = \frac{4}{(1+x^2+y^2)^2} dx \wedge dy$. *Therefore,* $\omega = \frac{1}{4}\omega_{FS}$ *and up to a choice of constant in the definition, both agree. See here: Cannas da Silva (2001).*

References: Cannas da Silva (2001)

Appendix B

Elements of Compact Lie Groups

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The following appendix sheds some light on basic results on the structure theory of Lie-algebras. Although the thesis does not use much of Lie theory, one can observe many results there follow from these observations as well. The following treatment is derived from Wang (2013). One important result in Lie-theory is that we have Riemannian Metrics on Lie-groups:

Theorem B.0.1 (Existence of Bi-invariant Riemannian Metrics)**.** *Consider a lie group G. There exists a Riemannian metric $\langle \cdot, \cdot \rangle$ on G, that is left and right g-invariant.*

Remark B.0.1. *For a proof of this result, see Wang (2013)*

Theorem B.0.2 (Existance of Complete Geodesics)**.** *If a Riemannian Manifold $(G, \langle \cdot, \cdot \rangle)$ is compact and connected, it has complete geodesics*

Remark B.0.2. *Let $x \in G$. Then for any $X \in T_xG$, we have that: $\exp_t : X \mapsto \exp(tX)$ being onto map, where $X \in T_xG$*

First, let's prove the following lemma:

Lemma B.0.3 (Decomposition of $\mathfrak{g}$)**.** *Let G be a Lie-group, and $\mathfrak{g}$ be its Lie-algebra, and $T \lhd G$ be the maximal toric subgroup of G. Let $\mathfrak{t}$ be the Lie-algebra of T. Then, we have the following decomposition theorem: $\mathfrak{g} = \bigcup_{g \in G} \operatorname{Ad}_g(\mathfrak{t})$*

Proof. By definition of the Ad_g action, this is equivalent to showing for any $X \in \mathfrak{g}$ that there exists $g' \in G$ such that $\operatorname{Ad}_{g'}(X) \in \mathfrak{t}$. Consider the bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on G. Note that:

$$\langle dL_g X, dL_g Y \rangle = \langle dR_g X, dR_g Y \rangle = \langle X, Y \rangle$$

Therefore, for $X, Y \in \mathfrak{g}$ and $g \in G$, we have that the metric is Ad-invariant:

$$\langle \operatorname{Ad}_g X, \operatorname{Ad}_g Y \rangle = \langle X, Y \rangle$$

Now, consider the continuous function f, defined on G: $f(g) = \langle Y, \operatorname{Ad}_g(X)\rangle$. This function takes its maximum value at $h_0 \in G$, so we now consider: $w(t) = \langle Y, \operatorname{Ad}_{\exp(tZ)} \operatorname{Ad}_{h_0} X\rangle$. Taking the derivative of w at $t = 0$, we have that:

$$\begin{aligned} \dot{w}(0) &= \langle Y, \operatorname{ad}_Z \operatorname{Ad}_{h_0} X\rangle && \text{(B.1)} \\ &= \langle Y, \operatorname{ad}_Z \operatorname{Ad}_{h_0} X\rangle = 0 && \text{(B.2)} \end{aligned}$$

Using the definition of $\operatorname{ad}_Y(Z) = -\operatorname{ad}_Z(Y) = [Y, Z]$:

$$\dot{w}(0) = \langle Y, \operatorname{ad}_{(\operatorname{Ad}_{h_0} X)} Z\rangle = 0$$

Now, since ad is the Lie bracket, it is skew-symmetric:

$$\dot{w}(0) = \langle \operatorname{ad}_{(\operatorname{Ad}_{h_0} X)} Y, Z\rangle = 0$$

As the dot product vanishes for all Z, we deduce that: $\operatorname{ad}_{(\operatorname{Ad}_{h_0} X)} Y = 0$. Therefore, $(\operatorname{Ad}_{h_0} X) \in \ker \operatorname{ad}(Y)$. However, we choose Y such that $\ker \operatorname{ad}(Y) = \mathfrak{t}$ since $\mathfrak{t}$ is a maximal abelian subalgebra. This concludes the proof of the lemma. $\square$

Remark B.0.3. *The following lemma allows us to prove that any Lie-group G is the union of conjugate tori $T \lhd G$.*

Theorem B.0.4 (Lie Group is Union of Conjugate Tori)**.** *Let G be a compact Lie group, and T a maximal torus of G. Then:*

$$G = \bigcup_{g \in G} \left(gTg^{-1}\right) \tag{B.3}$$

Proof. According to Lemma B.0.3: $\mathfrak{g} = \bigcup_{g \in G} \operatorname{Ad}_g(\mathfrak{t})$. Then, the result follows by integrating both sides, which we can do as G is compact. $\square$

Theorem B.0.5. *All maximal tori of G, $G_\mu = T_\mu \subset G$ are conjugate*

Proof. There exists a 1-1 correspondence between maximal tori in G and maximal abelian subalgebras in $\mathfrak{g}$ using the exponential map For maximality, we have that H is abelian if and only if $T_e H$ is abelian. Then we use Lie's Classification Theorem, that a maximal connected compact abelian subgroup is a torus. $\square$

Remark B.0.4. *Also, we note that the following is an elementary instructive exercise about the Ad_g invariance of the Lie bracket.*

Lemma B.0.6. *Let G be a compact Lie group and $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra, then any Cartan subalgebra $\mathfrak{t}'$ of $\mathfrak{g}$ is of the form $\operatorname{Ad}_g(\mathfrak{t})$ for some $g \in G$*

Proof. Then, from the above reasoning, we have if T is a maximal torus, then its Lie algebra $\mathfrak{t}$ is a maximal abelian subalgebra, and can therefore be written as:

$$\mathfrak{t}' = \mathrm{Ad}_g(\mathfrak{t}) \tag{B.4}$$

Integrating both sides via $t \mapsto \exp(tX)$ we have that:

$$T' = gTg^{-1} \tag{B.5}$$

$\square$

References: Wildberger (1992), Wang (2013)